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Planar $\mathcal{N}=4$ gauge theory and the Inozemtsev long range spin chain

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Abstract: We investigate whether the (planar, two complex scalar) dilatation operator of $\mathcal{N}=4$ gauge theory can be, perturbatively and, perhaps, non-perturbatively, described by an integrable long range spin chain with elliptic exchange interaction. Such a chain was introduced some time ago by Inozemtsev. In the limit of sufficiently "long" operators a Bethe ansatz exists, which we apply at the perturbative two- and three-loop level. Spectacular agreement is found with spinning string predictions of Frolov and Tseytlin for the two-loop energies of certain large charge operators. However, we then go on to show that the agreement between perturbative gauge theory and semi-classical string theory begins to break down, in a subtle fashion, at the three-loop level. This corroborates a recently found disagreement between three-loop gauge theory and near plane-wave string theory results, and quantitatively explains a previously obtained puzzling deviation between the string proposal and a numerical extrapolation of finite size three-loop anomalous dimensions. At four loops and beyond, we find that the Inozemtsev chain exhibits a generic breakdown of perturbative BMN scaling. Finally we suggest, assuming the AdS/CFT correspondence to be dynamically correct, that the observed three-loop disagreement between string and gauge theory might find an explanation as an order of limits (weak versus strong 't Hooft coupling) problem.

Keywords: AdS-CFT and dS-CFT Correspondence, Bethe Ansatz.

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1. Introduction and conclusions

Nearly two years ago a fresh approach [1] to uncover the dynamics of the AdS/CFT correspondence [2] was proposed. Progress in this direction is clearly required if we are to understand the gauge theory implications of the correspondence for quantum gravity and string theory, as well as the string theory implications of AdS/CFT for gauge theory. Berenstein, Maldacena and Nastase (BMN) [1] argued that the string spectrum on $AdS_5 \times S^5$ in a linearized (plane wave) limit, which is known [3], when expressed in planar $\mathcal{N} = 4$ gauge theory language, leads to a prediction for the all-loop exact anomalous dimensions of certain operators containing one large R-symmetry charge J. These predictions take the form of explicit functions in the BMN coupling constant

$$\lambda' = \frac{\lambda}{J^2}\,, \tag{1.1}$$

which are analytic around $\lambda' = 0$, with a finite radius of convergence. One can therefore expand them in integer power series in λ', and thus, so it seems, λ. This led BMN to suggest that the predictions could be checked, order by order, in *perturbative* planar $\mathcal{N} = 4$ gauge theory. The claim is that thereby one avoids a notorious AdS/CFT difficulty: the regime where AdS string theory is calculable corresponds to λ large, while gauge theoretic perturbation theory obviously assumes λ to be small. We will have more to say about this below.

The BMN proposal triggered a large number of interesting research papers, most of which cannot be discussed here (see the thorough reviews [4]). In particular, powerful techniques for the efficient computation of one-loop anomalous dimensions of $\mathcal{N}=4$ composite operators containing a large number of scalar fields were developed [5]. These activities led to the highly important insight of Minahan and Zarembo [6] that the *planar* one-loop anomalous dimensions of these operators can be found by diagonalizing the hamiltonian of an equivalent integrable spin chain by the Bethe ansatz method. In the case of just two complex scalars the spin chain is extremely simple, the Heisenberg XXX model. This is the "harmonic oscillator" of condensed matter theory. Similar spin chain techniques had been previously developed in the QCD context for a different class of operators, rather related to the space-time symmetries instead of the R-symmetry of the scalar operators [7]. A unified one-loop treatment of *all* conformal operators in the $\mathcal{N}=4$ theory was developed in [8], and shown to yield an integrable PSU$(2,2|4)$ super spin chain [9], as well as its corresponding Bethe ansatz.

The discovery of integrability in perturbative $\mathcal{N}=4$ theory is, potentially, of great importance. In a parallel development integrable structures were also pointed out on the string side of the correspondence [10]. Since the structures in gauge and string theory appear in very different regimes, the exciting prospect arises that the planar AdS/CFT system might be completely and non-perturbatively integrable. If true, this might allow to "exactly solve" planar $\mathcal{N}=4$ gauge theory, and thus free IIB string theory on $AdS_5 \times S^5$.

One way of looking at the BMN approach is to consider it a semi-classical limit [11] with an "artificial" large parameter, the charge J. In fact, a beautiful generalization of the BMN limit was proposed by Frolov and Tseytlin [12] (see also [13], and the closely related earlier work [14]). They consider rotating string solutions of the classical string equations of motion with *several* large charges $J_1, J_2 \ldots$. They then show that, firstly, quantum sigma-model corrections are suppressed by powers of the inverse total charge $J = J_1 + J_2 + \cdots$, and that, secondly, the obtained expressions for the string energies again expand in *integer* powers of the BMN coupling constant eq. (1.1). The latter fact led Frolov and Tseytlin to propose that, just as in the BMN case, the conjectured dual gauge theory operators should possess anomalous dimensions whose perturbative expansions match the just mentioned string energy expansions. An added benefit of this suggestion is that the symmetry charges J_i allow for a simple identification of those conformal $\mathcal{N}=4$ scaling operators which are natural candidates for the gauge theory analogs of the string theory states. The proposal was further elaborated on a large number of explicit examples [15, 16]. In all cases the classical problem can be exactly solved, the underlying reason being, once again, integrability [17] (classical, however, in this case).

The final (all-orders in λ') expressions for the classical string energies are rather non-trivial functions of the charge ratios J_i/J, as they are obtained by solving a non-linear system of equations. This is even true for the leading $\mathcal{O}(\lambda')$ term, which should correspond to a one-loop anomalous dimension in gauge theory. By applying the above mentioned Bethe ansatz to the relevant operators, it was indeed shown in a number of non-trivial examples that the leading string predictions can be reproduced by one-loop $\mathcal{N}=4$ gauge theory [18, 19, 20]. The relevant quantum spin chain is integrable since it possesses an infi-

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nite number of mutually commuting charges. As was recently shown in [21], these may be reproduced from the Bethe ansatz solutions as well, and precisely *match* the corresponding infinite tower of string sigma-model charges to leading order in λ'. For a recent, up-to-date review of "spinning strings", see [22].

The last result [21] would seem to be a near-proof, assuming the correctness of AdS/CFT, that (1) $\mathcal{N} = 4$ is integrable to *all* orders in λ', and that (2) the BMN and Frolov-Tseytlin (FT) proposals should indeed be valid to *all* orders in perturbation theory. The charges certainly are commuting for all values of λ' on the string side, and it would appear to be very difficult to imagine that there exist two inequivalent deformations of the matching leading order $\mathcal{O}(\lambda')$ commuting charges. Some caution is nevertheless warranted. Curious, unexpected and unaccounted structural agreements between small λ gauge theory and large λ string theory results have previously appeared in the context of AdS/CFT, see in particular [23]. One should definitely push perturbative $\mathcal{N} = 4$ to higher orders and see whether the agreement with BMN/FT persists.

The BMN prediction, originally checked in [1] at the one-loop level, was successfully tested in [24, 25] at two loops. It is usually stated in the recent literature that in [26] an all-orders proof was given. It appears to be difficult to rigorously justify some of the details of the proof, and we remain agnostic with regards to its validity. At any rate, while inspiring, the methodology of [26] is not of a constructive nature and we are unsure whether it does not, just like string theory, implicitly assume large λ.

In [27] a program was begun to derive the $\mathcal{N} = 4$ dilatation operator beyond the one-loop level. The two-loop correction for two complex scalars was found, and interpreted as a next-nearest neighbor deformation of the nearest neighbor Heisenberg spin chain. Excitingly, the result showed that the two-loop terms do not lift certain spectral planar one-loop degeneracies, which could be traced to the integrability property of [6]. This led to the conjecture [27] that the exact dilatation operator might correspond to an *integrable long-range spin chain*. Assuming integrability, and BMN scaling, the three-loop correction was derived. Using its explicit form, a numerical two- and three-loop gauge theory estimate for the FT prediction for the "folded string" [16] (see section 3 below) was obtained in [19]. It showed excellent agreement at two loops, but a strange 17% deviation at three loops. As a second sign of potential three-loop trouble, a string calculation of the leading $1/J$ correction to the BMN anomalous dimension formula was performed in [28] and, while beautifully agreeing at two loops, it *dis*agreed at three loops with [27].

Important further information comes from recent work by Beisert [29]. The method goes beyond [27] and uses, apart from field theoretic structural properties, symmetry arguments in order to constrain the three-loop dilatation operator (actually, for a larger class of fields than just two complex scalars). For our present purposes (we stick to the latter case in the present paper), the upshot of his study is as follows: (1) Perturbative integrability (in the sense of [27]) extends to at least three loops. (2) The two-loop dilatation operator is fixed up to one constant. This constant may be fitted to the known two-loop anomalous dimension of the Konishi field. This way one independently verifies the two-loop BMN prediction, in agreement with [24, 25]. (3) The three-loop dilatation operator is determined up to two unknown constants. These could be found if we knew two (independent) $\mathcal{N} = 4$

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anomalous dimensions (such as Konishi, plus one further field). However, no three-loop $\mathcal{N} = 4$ anomalous dimensions are rigorously known to date.[1] Therefore, the three-loop dilatation operator is not yet rigorously known. However, *if* we impose qualitative BMN scaling (i.e. if we assume in accordance with eq. (1.1) that the $\mathcal{O}(\lambda^3)$ anomalous dimensions of finite J BMN operators [30] scale like $\sim J^{-6}$), both constants are fixed and the infinite J BMN result is also quantitatively reproduced.[2]

In an interesting parallel development, it was established that a system closely related to full-fledged $\mathcal{N} = 4$ gauge theory, plane-wave matrix theory, also exhibits integrability to at least three loops [31]. This is a quantum mechanical system, and this result was rigorously derived without further assumptions.

Summarizing the last paragraphs, integrability seems to be a very stable concept in $\mathcal{N} = 4$ gauge theory even beyond one-loop; with further, indirect evidence coming from the string side of the AdS/CFT correspondence [10, 17, 21]. It is therefore important to gain deeper understanding of the integrable long range spin chain potentially describing planar $\mathcal{N} = 4$ gauge theory *non-perturbatively*. Such a spin chain, in contrast to the Heisenberg model, should contain an extra parameter, related to the Yang-Mills coupling constant. There exists one well-known integrable long-range spin chain, namely the Haldane-Shastry chain [32]. However, there the parameter corresponds to the length of the chain.[3] In particular, it is not possible to recover the one-loop Heisenberg model from Haldane-Shastry. There also exists a much less known integrable spin chain that precisely furnishes such an extra deformation parameter, the Inozemtsev long range spin chain [33]. Interestingly, it encompasses both the Haldane-Shastry chain and the Heisenberg chain. It is the purpose of this paper to investigate whether it might serve as a candidate for describing the sought all-loop dilatation operator (for two complex scalars). We shall find that this is indeed consistent with all currently known solid information. However, proving or disproving it will require more work.

The paper is organized as follows. In section 2 we describe the Inozemtsev spin chain, and show that it is indeed capable of emulating the $\mathcal{N} = 4$ dilatation operator up to at least three loops. It does not fix the yet to be determined two constants of the three-loop terms, as discussed above. This is not surprising, as they do not affect integrability [29]. We also describe a Bethe ansatz for this quantum chain, which is due to Inozemtsev as well (see [34], and references therein). This ansatz (unlike the full Inozemtsev spin chain) is limited to operators which are "long", which means that the interaction range is kept smaller than the length of the chain. We employ it to re-derive all known results [27] for finite J BMN operators. The Inozemtsev-Bethe ansatz is particularly interesting since it may, in principle, be used to derive non-perturbative results. In section 3 we apply the ansatz to the case of spinning strings. We are able to derive some of the highly involved *two-loop* expressions of [12, 16, 17] for folded and circular strings, in agreement with our

[1] It would clearly be of great importance to perform a reliable field theoretic computation of the planar three-loop anomalous dimensions of two independent fields in $\mathcal{N} = 4$ gauge theory.

[2] However, as we just mentioned, the $1/J$ corrections are *not* properly reproduced [28].

[3] This is reminiscent of the BMN coupling constant eq. (1.1), which also depends on the length of the spin chain. From our point of view, this is not an accident.

previous numerical study for a particular case [19]. However, at three loops we find that gauge theory yields a similar, but *different* result as compared to the FT proposal. On the positive side, we very accurately reproduce the previous numerical estimate for a special case [19].

Is this really bad news? Not necessarily. Of course, the perspicacious reader might suspect that the three-loop dilatation operator as conjectured in [27] and derived in [29] is simply wrong. In the absence of an independent, rigorous field theory computation, this remains a theoretical possibility; however, we currently do not see which mistake could have been made. On the other hand, it might be the case that the BMN and FT results are only valid at large λ (the one- and two-loop agreement might be due to some yet-to-be understood planar weak-strong coupling duality symmetry); the expansion is, after all, in λ', not λ. This issue came up already in certain puzzles related to the BMN limit [35]. In this context it is interesting that the Inozemtsev chain exhibits a generic breakdown of *perturbative* BMN scaling at *four* loops.[4] However, it is possible to recover BMN scaling non-perturbatively, i.e. at strong coupling (see section 2). A further, more troubling explanation for the deviation we are finding would be a large-N phase transition when we go from weak to strong coupling [36].

In this paper we have shown how to extend the Bethe ansatz for planar one-loop $\mathcal{N}=4$ gauge theory to higher loops. Our procedure is completely sound up to three loops. Of course we are far from proving that our proposal is valid beyond three loops, let alone non-perturbatively. However, having identified an integrable long-range spin chain which properly describes the dilatation operator up to three loops, we should take it seriously. Integrability is a subtle and tight structure, and not found abundantly.

We have not been able to explicitly mention in this introduction many interesting, closely related recent contributions to this subject of study, see in particular [37].

2. The Inozemtsev spin chain and $\mathcal{N}=4$ gauge theory

2.1 Long-range spin chains with elliptic and hyperbolic interactions

More than one decade ago, Inozemtsev [33] proposed a spin chain with long range interaction which interpolates between the Heisenberg and Haldane-Shastry [32] spin chains. His hamiltonian takes the form

$$I=\sum_{j=1}^{L}\sum_{n=1}^{L-1}\mathcal{P}_{L,\pi/\kappa}(n)(1-P_{j,j+n})\,, \tag{2.1}$$

[4] We note in passing that we suspect the Inozemtsev model to be equally capable of describing the plane-wave matrix model mentioned above, which has also been shown to be "perturbatively" integrable to at least three loops [31]. There BMN scaling may be recovered, up to three loops, by a redefinition of the mass parameter of the model. If our Inozemtsev ansatz is correct we suspect that such a redefinition is no longer possible at the four-loop level (or else, integrability of the model should break down). This might be a more feasible check of our Inozemtsev ansatz as opposed to considering four-loop perturbation theory in full-fledged $\mathcal{N}=4$ field theory. In turn, if the plane wave mechanics stays integrable at four loops while still being consistent, after redefinition, with BMN scaling, it would strongly suggest that there exist further integrable long range spin chains differing from Inozemtsev.

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where the operator P_{jk} permutes the spins at sites i and j, and the interaction strength is defined in terms of the elliptic Weierstrass function $\mathcal{P}_{L,\pi/\kappa}(n)$ with periods L and $i\pi/\kappa$. We recall that the Weierstrass function can be defined by the following series

$$\mathcal{P}_{L,\pi/\kappa}(z) = \frac{1}{z^2} + \sum_{m,n}{}' \left(\frac{1}{(z - mL - in\pi/\kappa)^2} - \frac{1}{(mL + in\pi/\kappa)^2} \right),$$

where the prime means that the term with $m = n = 0$ is omitted. When one of the two periods becomes infinite, the Weierstrass function becomes a trigonometric or hyperbolic function. If $\kappa \to 0$, this function goes smoothly to the Haldane-Shastry interaction,

$$\lim_{\kappa \to 0} \mathcal{P}_{L,\pi/\kappa}(z) = \left(\frac{\pi}{L}\right)^2 \left(\frac{1}{\sin^2 \pi z/L} - \frac{1}{3} \right),$$

while for $L \to \infty$ the interaction takes the form of a hyperbolic function, and decays exponentially with the distance,

$$\lim_{L \to \infty} \mathcal{P}_{L,\pi/\kappa}(z) = \kappa^2 \left(\frac{1}{\sinh^2 \kappa z} + \frac{1}{3} \right).$$

The Heisenberg limit can be recovered in the limit $\kappa \to \infty$, where the imaginary period vanishes. To be able to properly take this limit we define the "coupling constant"

$$t \equiv e^{-2\kappa}, \tag{2.2}$$

and we use a redefined interaction strength, by removing an additive and a multiplicative constant

$$h_{t,L}(n) \equiv \frac{\mathcal{P}_{L,\pi/\kappa}(n)}{4\kappa^2} - \frac{1}{12} = \sum_{m=-\infty}^{\infty} \frac{t^{n+Lm}}{(t^n - t^{Lm})^2}.$$

The Heisenberg interaction can be obtained to leading order in t of the Inozemtsev interaction

$$\lim_{t \to 0} \frac{h_{t,L}(n)}{t} = \delta_{n,1} + \delta_{n,L-1}.$$

Going beyond the first order in t, we notice that $h_{t,L}(n)$ starts as t^n (plus higher orders in t), as long as $n < L/2$, therefore the Inozemtsev interaction involves spins separated by a distance of at most n at order n in t. This property agrees with the structure of the perturbative dilatation operator in the planar limit [24].

Inozemtsev performed a detailed study of the hamiltonian (2.1) (see [34] for a review) and gave convincing arguments in the favor of its integrability, by finding the corresponding Lax pair.[5] In addition, he explicitly constructed the first of the higher charges commuting

[5] According to Inozemtsev [34], who invokes a result by Krichever [38], this is the most general Lax pair corresponding to a spin chain with long range interaction and pairwise exchange given by permutation operator. However, his arguments do not directly apply to the case when the interaction involves more than two spins, and we should keep an open mind as far as concerns the existence of integrable long range spin chains of a more general type than Inozemtsev. For these putative chains, the subtraction of higher charges from the lowest charge (i.e. the hamiltonian) would not allow one to eliminate interactions involving more than two spins.

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with the hamiltonian. The existence of this first higher charge usually indicates complete integrability for spin chains. However, the general form of the commuting charges or the monodromy matrix are still out of reach. The connection with the Calogero-Moser hamiltonian with elliptic potential presented in [34] may help finding the exact solution of this chain. Among the aforementioned limiting cases, the Heisenberg case $\kappa \to \infty$ and the Haldane-Shastry case, $\kappa \to 0$ are completely solved, the first using the Bethe ansatz and the second by exploiting the Yangian symmetry [39].

In the following, we are going to concentrate on the hyperbolic case, where the length of the chain becomes infinite (i.e. $L \to \infty$), and the interaction takes the simpler form

$$h_t(n) = \frac{1}{4\sinh^2 \kappa n} = \frac{t^n}{(t^n-1)^2}\,,$$

that is, the periodicity effects of the interaction are negligible. This case is easier to treat analytically, but still quite interesting from the point of view of comparison with the known results for the $\mathcal{N}=4$ SYM theory. In particular, it allows to study BMN operators at finite (but large) $J \simeq L$. The full elliptic potential takes into account the effects related to the periodicity of the chain, therefore one could hope that it contains the information concerning "short" operators such as Konishi.

In the hyperbolic regime we can use an analytic continuation for the monodromy matrix and for the conserved charges of the Haldane-Shastry spin chain [39, 40, 41]. The Haldane-Shastry conserved charges still commute (and commute with the Yangian generators) if one replaces the coordinates $z_k = e^{2\pi i k/L}$ by $z_k = t^{-k}$ [40].[6] We recall that the Yangian generators are given by [39]

$$\begin{aligned} Q_0^a &= \sum_j \sigma_j^a\,, \\ Q_1^a &= i\varepsilon^{abc} \sum_{j<k} \frac{z_j+z_k}{z_j-z_k}\,\sigma_j^b\,\sigma_k^c\,. \end{aligned} \tag{2.3}$$

Let us emphasize that the Yangian is a true symmetry (i.e. it commutes with the conserved charges) only for the hyperbolic chain, when the length of the chain is infinite. Otherwise, like in the case of Heisenberg spin chain [42], in the full elliptic case the Yangian symmetry is broken by "boundary terms". The higher conserved charges of the hyperbolic spin model are also obtained by simply replacing $z_k = t^{-k}$ in the expression for the Haldane-Shastry conserved charges [39, 41]

$$I_2 = \sum_{ij}{}' \frac{z_i z_j}{z_{ij} z_{ji}}(P_{ij}-1)\,,$$

[6] The generating function for the conserved charges of the Haldane-Shastry hamiltonian was identified in [41] by taking the limit $\lambda \to \infty$ of the monodromy matrix for the Calogero-Sutherland model with spin. The same argument can be easily applied to the hyperbolic case, since in this case the leading term of the quantum determinant is trivial, $\Delta_0(u) = 0$, as well as its derivatives. It follows that the next-to-leading term $\Delta_1(u)$ in λ^{-1} in the expansion of the quantum determinant can be used to generate the commuting charges in exactly the same fashion as for Haldane-Shastry spin chain.

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$$I_3 = \sum_{ijk}{}' \frac{z_i z_j z_k}{z_{ij} z_{jk} z_{ki}} (P_{ijk} - 1) \, ,$$

$$I_4 = \sum_{ijkl}{}' \frac{z_i z_j z_k z_l}{z_{ij} z_{jk} z_{kl} z_{li}} (P_{ijkl} - 1) - 2 \sum_{ij}{}' \left(\frac{z_i z_j}{z_{ij} z_{ji}} \right)^2 (P_{ij} - 1) \, , \tag{2.4}$$

where the prime indicates that the sum is over distinct summation indices, $z_{ij} = z_i - z_j$, and P_{ijk}, etc. represent cyclic permutations. A systematic procedure to find all the conserved charges was given in [41], although the explicit construction of these charges becomes more and more involved at higher orders.

The diagonalization of the conserved charges is achieved, in the case of nearest-neighbor interaction, by the algebraic Bethe ansatz. In the case of the long-range interaction, the algebraic Bethe ansatz does not work any more, just like in the case of the Haldane-Shastry spin chain. There, the diagonalization can be performed by using the Yangian symmetry and the relation to the Calogero-Sutherland model [39, 43]. In the hyperbolic case it is still possible to use the so-called asymptotic Bethe ansatz, valid in the regime where L is very large but still finite. The phase acquired by a quasi-particle (magnon) traveling around the circumference of the chain is given by the phase gained by its scattering with all the other magnons in the system

$$\exp\left(i p_j L\right) = \exp\left(i \sum_{\substack{k=1 \\ k \neq j}}^{M} \chi(p_j, p_k) \right) , \tag{2.5}$$

where $\chi(p_j, p_k)$ is the phase shift for the scattering of two magnons of momenta p_j and p_k and M is the number of magnons. Inozemtsev computed the phase shift for the scattering of two magnons,

$$\cot \frac{\chi(p_j, p_k)}{2} = \varphi(p_j) - \varphi(p_k) \, ,$$

$$\varphi(p) = \frac{p}{2\pi i \kappa} \zeta_1\left(\frac{i\pi}{2\kappa}\right) - \frac{1}{2i\kappa} \zeta_1\left(\frac{ip}{2\kappa}\right) , \tag{2.6}$$

as well as the magnon energy,

$$\varepsilon(p) = \frac{1}{i\pi\kappa} \zeta_1\left(\frac{i\pi}{2\kappa}\right) - \frac{1}{4\kappa^2} \mathcal{P}_1\left(\frac{ip}{2\kappa}\right) - \varphi^2(p) \, , \tag{2.7}$$

where $\mathcal{P}_1(z) = \mathrm{d}\zeta_1(z)/\mathrm{d}z$ denotes the Weierstrass elliptic functions of periods 1 and $i\pi/\kappa$, and the quasi-periodic zeta function is odd, $\zeta_1(-z) = -\zeta_1(z)$. The function $\varphi(p)$ is also quasi-periodic, $\varphi(p + 2\pi) = \varphi(p)$ and $\varphi(p + 2i\kappa) = \varphi(p) - i$. While the phase shift and magnon energy (2.6), (2.7) are exact, the Bethe ansatz formula (2.5) is asymptotic, i.e. it is valid for chains of finite but large length L. It is easy to check that in the limit $\kappa \to \infty$, the magnon energy and the phase shift reduce to the corresponding Heisenberg values, in particular $\varphi(p) \to (1/2)\cot(p/2)$.

2.2 Comparing the conserved charges of the Inozemtsev spin chain to the dilatation operator

If we assume that the dilatation operator in the $su(2)$ sub-sector of $\mathcal{N}=4$ theory is integrable to all orders in perturbation theory, as first conjectured in [27], it should correspond to some integrable long-range spin chain. We believe that the Inozemtsev spin chain is an interesting and natural candidate. As we shall prove, it is definitely capable of describing the dilatation operator in the subsector to at least three loops. However, if the description extends to four loops and beyond, perturbative BMN scaling would have to be violated at the four loop level.

If the dilatation operator contains, beyond two loops, interaction terms which involve more than two spins, the Inozemtsev hamiltonian is not, by itself, sufficient to reproduce the former.[7] In this case, one should consider a linear combination of the even parity conserved charges of the Inozemtsev spin chain.[8] The most general expression for the dilatation operator will then be

$$D(\lambda) = L + f_1(\lambda) I_2(t) + f_2(\lambda) I_4(t) + f_3(\lambda) I_6(t) + \cdots \tag{2.8}$$

where $t = t(\lambda)$ is such that $\lim_{\lambda\to 0} t/\lambda = 1/16\pi^2$, $f_k(\lambda)$ are functions regular at $\lambda = 0$ and $I_{2k}(t)$ are the even parity conserved charges of Inozemtsev spin chain.

To determine the precise relation between the 't Hooft coupling constant λ and the parameter t, as well as the functions $f_k(\lambda)$, we have to use arguments going beyond perturbation theory. The first example of such an argument concerns the behavior of the rapidity $\varphi(p)$ in the BMN limit, where the coupling constant λ and the chain length L are both large, $\lambda = \lambda' L^2$. Since it enters the Bethe ansatz equations, and therefore all the observables, the combination $\varphi(p)/L$ should have a well-behaved BMN limit. The rapidity (2.6) can be written as an infinite sum

$$\begin{aligned}\varphi(p) &= \frac{1}{2}\cot\frac{p}{2} + \frac{1}{2}\sum_{n>0}\left[\cot\left(\frac{p}{2} - i\kappa n\right) + \cot\left(\frac{p}{2} + i\kappa n\right)\right] \\ &= \frac{1}{2}\cot\frac{p}{2} + 2\sum_{n>0}\frac{t^n \sin p}{(1-t^n)^2 + 4t^n\sin^2(p/2)}\,.\end{aligned} \tag{2.9}$$

where in the second line we have used the perturbative parameter t instead of the parameter κ. Expanding this result to the first few orders in p we obtain

$$\varphi(p) = \frac{1}{2}\cot\frac{p}{2} + 2p\sum_{n>0}\frac{t^n}{(1-t^n)^2} - \frac{p^3}{3}\sum_{n>0}\frac{t^n(1+4t^n+t^{2n})}{(1-t^n)^4} + \cdots\,.$$

[7] The three-loop dilatation operator proposed in [27] does contain four-spin terms, as these are necessary for perturbative BMN scaling. If we drop this requirement, the yet undetermined two constants [29] discussed in the introduction may be chosen so as to eliminate the four spin terms. The future will tell.

[8] In [27] it was argued that only connected planar diagrams contribute to the dilatation operator (see also [24]). A completely rigorous all-orders proof is not easy due to the known problems with regularization by dimensional reduction at large loop order in $\mathcal{N}=4$ gauge theory. Non-connected diagrams would require non-linear combinations of the conserved charges.

In the BMN limit, the leading behavior of the magnon momentum is $p \sim 1/L$, and therefore the rapidity scales as $\varphi \sim 1/p \sim L$. In order to obtain a finite but non-zero result for the rescaled rapidity φ/L in the perturbative regime $\lambda \sim 0$ where $\kappa \to \infty$ (i.e. $t \to 0$) we see that we need to identify

$$\frac{\lambda}{16\pi^2} = \sum_{n>0} \frac{t^n}{(1-t^n)^2} = \sum_{n>0} \frac{1}{4\sinh^2(\kappa n)} \,. \tag{2.10}$$

On then finds that

$$\begin{aligned} \frac{\varphi}{L} &= \frac{1}{Lp} + \frac{2}{L^2}\frac{\lambda}{16\pi^2}(Lp) - \frac{1}{3L^4}\left[\frac{\lambda}{16\pi^2} + 6\left(\frac{\lambda}{16\pi^2}\right)^2 - 12\left(\frac{\lambda}{16\pi^2}\right)^3\right](Lp)^3 + \cdots \\ &= \frac{1}{Lp} + \frac{\lambda'}{8\pi^2}(Lp) - \frac{\lambda'^2}{128\pi^4}(Lp)^3 + L^2\,\frac{\lambda'^3}{1024\pi^6}(Lp)^3 + \cdots \qquad \text{as } L \to \infty \end{aligned} \tag{2.11}$$

where we have used the finite BMN coupling constant $\lambda' = \lambda/L^2$.

We see that at $\mathcal{O}(\lambda'^3)$ a quadratic divergence appears in the perturbative expression for the rescaled rapidity φ/L. The consequences of this perturbative scaling violation will be discussed in section 2.4 below.

It is interesting to see that a finite rescaled rapidity φ/L may be obtained *provided* we stay in the strong-coupling regime where $\kappa \to 0$ (i.e. $t \to 1$) and thus $\lambda = 2\pi^4/3\kappa^2 \to \infty$. Here, it is useful to use a dual representation for the rapidity $\varphi(p)$

$$\varphi(p) = \frac{\pi}{2\kappa}\coth\frac{\pi p}{2\kappa} + \frac{\pi}{2\kappa}\sum_{m>0}\left[\coth\frac{\pi(p-2\pi m)}{2\kappa} + \coth\frac{\pi(p+2\pi m)}{2\kappa}\right] - \frac{p}{2\kappa}\,, \tag{2.12}$$

where the last term insures the desired quasi-periodicity properties, $\varphi(p+2\pi) = \varphi(p)$ and $\varphi(p+2i\kappa) = \varphi(p) - i$. The terms with $m > 0$ form a power series in the dual parameter $e^{-2\pi^2/\kappa}$ and they vanish exponentially in the BMN limit, since in this case $1/\kappa \sim L \to \infty$. The remaining part obviously obeys BMN scaling and may be written as

$$\varphi(p) = \frac{1}{2\pi}\sqrt{\frac{3}{2}\lambda'}\coth\frac{p}{2\pi}\sqrt{\frac{3}{2}\lambda'}\,, \tag{2.13}$$

where we have replaced $\varphi(p)/L$ by $\varphi(p)$ and pL by p.

In principle, the explicit knowledge of all the conserved charges of the Inozemtsev chain and of their eigenvalues would allow to obtain information about the functions $f_k(\lambda)$ in equation (2.8). Since, at least for the moment, we lack this knowledge we have to restrict ourselves to comparing the relation (2.8) to the perturbative results which are already known [27, 29]. Let us then compare the expression (2.8) to the dilatation operator,

$$D(\lambda) = L + \sum_{k>0}\left(\frac{\lambda}{16\pi^2}\right)^k D_{2k}$$

as stated in [27]. Rewritten using commuting permutations, the contributions to the dilatation operator up to three loops are

$$
\begin{aligned}
D_0 &= L\,, \\
D_2 &= 2L - 2\sum_i P_{i,i+1}\,, \\
D_4 &= -6L + 8\sum_i P_{i,i+1} - 2\sum_i P_{i,i+2}\,, \\
D_6 &= 40L - 56\sum_i P_{i,i+1} + 16\sum_i P_{i,i+2} - 4\sum_i (P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3})
\end{aligned} \tag{2.14}
$$

We recall that D_2 was obtained by diagrammatic computation in $\mathcal{N}=4$ SYM theory and the expression of D_4 was checked in several ways [24, 25]. The expressions for D_6 were obtained initially [27, 44] supposing perturbative integrability and imposing perturbative BMN scaling. The expression for D_6 was subject to controversies, since it disagrees with the results from the near plane-wave string theory [28]. Later on, Beisert [29] checked that the integrability of D_6 is insured by supersymmetry, and that the two free coefficients are fixed if one imposes perturbative BMN scaling.

The conserved charges of the Inozemtsev spin chain can be expanded in series of t, and therefore of the coupling constant λ. By inverting the series (2.10) we obtain

$$
t = \frac{\lambda}{16\pi^2} - 3\left(\frac{\lambda}{16\pi^2}\right)^2 + 14\left(\frac{\lambda}{16\pi^2}\right)^3 + \cdots\,, \tag{2.15}
$$

Inserting this expression into (2.4) and expanding in powers of λ we get

$$
I_{2k} = \sum_{n>0}\left(\frac{\lambda}{16\pi^2}\right)^n I_{2k}^{(n)}\,.
$$

The explicit expression for the first few orders in the hamiltonian is

$$
\begin{aligned}
I_2^{(1)} &= 2L - 2\sum_i P_{i,i+1} \\
I_2^{(2)} &= 2\sum_i P_{i,i+1} - 2\sum_i P_{i,i+2}\,, \\
I_2^{(3)} &= -10\sum_i P_{i,i+1} + 12\sum_i P_{i,i+2} - 2\sum_i P_{i,i+3}\,.
\end{aligned}
$$

In the following, we use a redefinition of the fourth conserved charge, $\tilde{I}_4(\lambda) = I_4(\lambda) - \frac{\lambda}{8\pi^2}I_2(\lambda)$. This is done for the purpose of having the fourth conserved charge of the Heisenberg chain as the first non-trivial order in the λ expansion

$$
\begin{aligned}
\tilde{I}_4^{(1)} &= \tilde{I}_4^{(2)} = 0\,, \\
\tilde{I}_4^{(3)} &= -8L + 8\sum_i P_{i,i+1} - 4\sum_i P_{i,i+2} + 4\sum_i P_{i,i+3} - \\
&\quad -8\sum_i\left(P_{i,i+3}P_{i+1,i+2} - P_{i,i+2}P_{i+1,i+3}\right)\,,
\end{aligned}
$$

It is not obvious that a similar redefinition can be done for all the conserved charges, but at least we know that we can arrange to have $\tilde{I}_6^{(3)} = 0$.

We can now reconstitute the perturbative dilatation operator (2.14) using the Inozemtsev conserved charges

$$\begin{aligned} D_2 &= I_2^{(1)}\,, \\ D_4 &= I_2^{(2)} - 3I_2^{(1)}\,, \\ D_6 &= I_2^{(3)} + \frac{1}{2}\tilde{I}_4^{(3)} - 3I_2^{(2)} + 22I_2^{(1)}\,. \end{aligned} \tag{2.16}$$

If we reinstate the constants which were fixed by imposing perturbative BMN scaling [27], the most general form for the third loop dilatation operator allowed by perturbative integrability is

$$D_6 = I_2^{(3)} + a_1\tilde{I}_4^{(3)} - 3I_2^{(2)} + a_2 I_2^{(1)}\,. \tag{2.17}$$

Comparing to the expression (2.8), we conclude that $f_1(\lambda) = 1 - (2a_1+3)\frac{\lambda}{16\pi^2} + \cdots$ and $f_2(\lambda) = a_1 + \cdots$, the constant a_2 appearing at next order in λ. In conclusion, we can reproduce the dilatation operator up to three loops using the conserved charges of the Inozemtsev spin chain.

2.3 Perturbative results from the asymptotic Bethe ansatz. Dimensions of the BMN operators

In the following, we are going to compute perturbatively the dimensions of the BMN operators up to third order, supposing that the dilatation operator is integrable and its integrability is governed by the Inozemtsev spin chain. The strategy is to compute the magnon momenta by using the asymptotic Bethe ansatz equation (2.5) and to substitute them into the expression of the dilatation operator known from perturbative calculations.

The asymptotic Bethe ansatz equations have the form (2.5), which we can rewrite as

$$e^{ip_jL} = \prod_{\substack{k=1\\k\neq j}}^{M} \frac{\varphi_j - \varphi_k + i}{\varphi_j - \varphi_k - i}\,, \tag{2.18}$$

where M is the magnon number and the rapidity $\varphi_j = \varphi(p_j)$ is given by the expression (2.9). To see the connection with the usual Bethe ansatz for the Heisenberg chain, we can replace the momentum by a rapidity variable $u_j = \frac{1}{2}\cot\frac{p_j}{2}$ such that the left hand side of eq. (2.18) becomes

$$e^{ip_jL} = \left(\frac{u_j + i/2}{u_j - i/2}\right)^L,$$

while it is clear from eq. (2.9) that $\varphi_j = u_j + \mathcal{O}(t)$. More precisely, up to third order in t we have

$$\varphi(p) = u(p) + 2t\sin p + 2t^2\sin p\,(1+2\cos p) + 8t^3\sin p\cos^2 p + \cdots\,.$$

We can perturbatively solve the Bethe ansatz equation for the two-magnon case, with the two magnon momenta satisfying $p_1 = -p_2 = p$. Since we will compare our results to the anomalous dimensions of the BMN operators

$$\mathrm{Tr}\, Z^J\Phi^2 + \cdots\,,$$

we are going to use $J+2$ to denote the length of the chain,

$$L = J + 2\,.$$

The Heisenberg solution for the magnon momentum is

$$p = p_n \equiv \frac{2\pi n}{J+1}\,,$$

and the corrections to this value are given by the equation

$$\begin{aligned} p = p_n &- \frac{8t\sin p\sin^2\frac{p}{2}}{J+1} - \frac{8t^2\sin p\sin^2\frac{p}{2}}{J+1}\left(8\cos^4\frac{p}{2} - 4\cos^2\frac{p}{2} + 1\right) - \\ &- \frac{32t^3\sin p\,\sin^2\frac{p}{2}}{3(J+1)}\left(64\cos^8\frac{p}{2} - 96\cos^6\frac{p}{2} + 48\cos^4\frac{p}{2} - 16\cos^2\frac{p}{2} + 3\right) + \cdots\,. \end{aligned}$$

The perturbative solution of the Bethe ansatz equation up to third order in t is

$$\begin{aligned} p = p_n &- \frac{8t\sin p_n\sin^2\frac{p_n}{2}}{J+1} + \\ &+ \frac{64t^2\,\sin p_n\sin^4\frac{p_n}{2}\left(4\cos^2\frac{p_n}{2} - 1\right)}{(J+1)^2} - \frac{8t^2\sin p_n\,\sin^2\frac{p_n}{2}\left(8\,\cos^4\frac{p_n}{2} - 4\,\cos^2\frac{p_n}{2} - 1\right)}{J+1} - \\ &- \frac{512t^3\,\sin p_n\sin^6\left(\frac{p_n}{2}\right)\left(24\cos^4\frac{p_n}{2} - 13\cos^2\frac{p_n}{2} + 1\right)}{(J+1)^3} + \\ &+ \frac{128\,t^3\sin p_n\sin^4\left(\frac{p_n}{2}\right)\left(48\cos^6\frac{p_n}{2} - 44\cos^4\frac{p_n}{2} + 4\cos^2\frac{p_n}{2} + 1\right)}{(J+1)^2} - \\ &- \frac{32t^3\sin p_n\,\sin^2\frac{p_n}{2}\left(64\cos^8\frac{p_n}{2} - 96\cos^6\frac{p_n}{2} + 48\cos^4\frac{p_n}{2} - 16\cos^2\frac{p_n}{2} + 3\right)}{3(J+1)} + \cdots\,. \end{aligned}$$

Using the perturbative expansion (2.15) for t in powers of λ, we can transform this expression into a series in λ. Then, we insert the solution for the momentum in the eigenvalue formula for the dilatation operator, which is a consequence of (2.14),

$$E(\lambda) = J + 2\left(1 + \frac{\lambda}{2\pi^2}\sin^2\frac{p}{2} - \frac{\lambda^2}{8\pi^4}\sin^4\frac{p}{2} + \frac{\lambda^3}{16\pi^6}\sin^6\frac{p}{2} + \cdots\right).$$

Using `Maple` to perform the resulting series expansion, we checked that the two-magnon energy at order k in λ, $E^{(k)}$, has the behavior predicted in [27], up to order $k=3$

$$\begin{aligned} E^{(1)} &= \frac{1}{\pi^2}\sin^2\frac{p_n}{2}\,, \\ E^{(2)} &= -\frac{1}{\pi^4}\sin^4\frac{p_n}{2}\left(\frac{1}{4} + \frac{1}{J+1}\cos^2\frac{p_n}{2}\right), \\ E^{(3)} &= \frac{1}{\pi^6}\sin^6\frac{p_n}{2}\left(\frac{1}{8} + \frac{\frac{1}{2}\cos^4\frac{p_n}{2} + \frac{3}{4}\cos^2\frac{p_n}{2}}{J+1} + \frac{\frac{5}{2}\cos^4\frac{p_n}{2} - \frac{3}{4}\cos^2\frac{p_n}{2}}{(J+1)^2}\right). \end{aligned} \tag{2.19}$$

The agreement with the perturbative results in gauge theory is perfect, up to three loops, as far as the anomalous dimensions of the large J BMN operators are concerned. For completeness, let us look at one more order and, in particular, to compare with the conjectured results of [44].

2.4 Fourth order dilatation operator and the breakdown of perturbative BMN scaling

The four loop dilatation of $\mathcal{N} = 4$ gauge theory is not yet rigorously known. In [44] a proposal for its structure was put forward, based (1) on extending the planar pair symmetry, due to integrability [27], and (2) on insisting on *perturbative* BMN scaling. As we shall see below, this proposal will turn out to be inconsistent with the Inozemtsev integrable structure. It might be that the constraints coming from all-order integrability become tighter at higher orders and that it will become impossible to satisfy at once integrability and perturbative BMN scaling. However, this does not mean automatically that [44] is wrong. Indeed, it is not excluded that several integrable long-range spin chains exist (see the comments in footnotes 4 and 5).

The proposal in [44] can be rewritten as

$$\begin{aligned}
D_8 =\ & 222L - 648\sum_i P_{i,i+1} + \frac{1340}{3}\sum_i P_{i,i+2} - \frac{1796}{3}\sum_i P_{i,i+3} + 6\sum_i P_{i,i+4} + \\
& + \frac{842}{3}\sum_i P_{i,i+1}P_{i+2,i+3} + \frac{1972}{3}\sum_i P_{i,i+3}P_{i+1,i+2} - 366\sum_i P_{i,i+2}P_{i+1,i+3} + \\
& + \frac{16}{3}\sum_i (P_{i,i+3}P_{i+2,i+4} + P_{i,i+2}P_{i+1,i+4} - P_{i,i+4}P_{i+2,i+3} - P_{i,i+4}P_{i+1,i+2} - \\
& \qquad - P_{i,i+4}P_{i+1,i+3} + P_{i,i+3}P_{i+1,i+4}) \,,
\end{aligned} \tag{2.20}$$

where we have chosen the value $\beta = -428/3$ in the expression of [44], since this value allows to reconstitute the term in parenthesis, which is a very natural combination of permutation. We recall that β corresponds in [44] to a unitary transformation of D_8 and that it does not affect the eigenvalue of D_8.

We also expand the first two even Inozemtsev conserved charges to fourth order in λ,

$$\begin{aligned}
I_2^{(4)} =\ & 62\sum_i P_{i,i+1} - 78\sum_i P_{i,i+2} + 18\sum_i P_{i,i+3} - 2\sum_i P_{i,i+4} \,, \\
\tilde{I}_4^{(4)} =\ & 48L - 36\sum_i P_{i,i+1} + 16\sum_i P_{i,i+2} - 40\sum_i P_{i,i+3} + 12\sum_i P_{i,i+4} - \\
& -8\sum_i P_{i,i+1}P_{i+2,i+3} + 56\sum_i P_{i,i+3}P_{i+1,i+2} - 48\sum_i P_{i,i+2}P_{i+1,i+3} + \\
& +8\sum_i (P_{i,i+3}P_{i+2,i+4} + P_{i,i+2}P_{i+1,i+4} - P_{i,i+4}P_{i+2,i+3} - P_{i,i+4}P_{i+1,i+2} - \\
& \qquad - P_{i,i+4}P_{i+1,i+3} + P_{i,i+3}P_{i+1,i+4}) \,.
\end{aligned}$$

We see that ansatz (2.20) can be written as

$$D_8 = I_2^{(4)} + \frac{2}{3}\tilde{I}_4^{(4)} + \text{lower order terms} \,, \tag{2.21}$$

where "lower order terms" stands for a term of the form $\sum_i (P_{i,i+1}P_{i+2,i+3} + P_{i,i+3}P_{i+1,i+2})$ which we expect to come from $\tilde{I}_6^{(4)}$, plus a linear combination of $\tilde{I}_4^{(3)}$, $I_2^{(3)}$, $I_2^{(2)}$, $I_2^{(1)}$.

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What we learn from this exercise is that, imposing the integrability perturbatively, as it was done in [27, 44], we obtain the building blocks of the Inozemtsev spin chain (modulo the contribution of the sixth conserved charge, for which we do not have a quantitative estimate). However, the coefficients multiplying $I_{2k}^{(n)}$ in (2.16) disagree with the all-order expression (2.8). The expression (2.8) implies that the coefficient of $I_{2k}^{(n)}$ in D_{2n} should not depend on n. This is not the case for $I_4^{(3)}$ and $I_4^{(4)}$ appearing in D_6 and D_8 in equations (2.16), (2.21). In order to be compatible with the proposal (2.8), the fourth order dilatation operator has to be

$$\tilde{D}_8 = I_2^{(4)} - 3I_2^{(3)} + a_2 I_2^{(2)} + b_1 I_2^{(1)} + a_1 \tilde{I}_4^{(4)} + b_2 \tilde{I}_4^{(3)} + b_3 \tilde{I}_6^{(4)} \,. \tag{2.22}$$

A possible explanation of the mismatch is related to the fact that the coefficients in D_6 and D_8 which were left free by perturbative integrability were fixed imposing perturbative BMN scaling [27, 44]. However, in (2.22) there are three free constants left, once we have fixed, at third order, a_1 and a_2, while Beisert obtains [44] six constants to be fixed by perturbative BMN. This point certainly deserves a better understanding.

Independently of the previous analysis, we know that the Inozemtsev spin chain cannot reproduce, starting from the four-loop order, an anomalous dimension which obeys BMN scaling. The reason is our finding in eq. (2.11): The rescaled rapidity $\varphi(p)/L$ does not have an expansion in λ consistent with perturbative BMN scaling. The first violation appears at third order in λ, and therefore will contribute to the correction to the two magnon energy at the *fourth order*.

It is interesting that the BMN scaling of the rapidity, which breaks down in the Inozemtsev model at *weak coupling* (i.e. when λ is small), is nevertheless recovered at *strong coupling* (i.e. when λ is large). In particular, recall that in the calculable regimes of string theory λ is certainly large, and our finding shows, quite independently of the question of the suitability of the Inozemtsev model for the gauge theory dilatation operator beyond three loops, that there *can* be a weak coupling-strong coupling difference in a given model as far as the thermodynamic scaling properties are concerned. Indeed, (2.9) and (2.12) are expansions of the same rapidity $\varphi(p)$. The former equation (2.9) is suitable for small λ and is inconsistent with scaling. The latter equation (2.12) is appropriate for large λ, and exhibits BMN scaling. Technically, this happens because at large λ we neglect terms of the type $e^{-\sqrt{\lambda}}$ in (2.12), which *resum* when we go to weak coupling, leading to non-zero terms at $\lambda = 0$ which break the thermodynamic scaling.

Of course it is currently not clear whether these observations, apart from pointing out various potential subtleties related to scaling, are directly relevant to the non-perturbative gauge theory. Proving or disproving BMN scaling in the latter is very hard (in fact, much harder than proving integrability!), and has not even been achieved at three loops. Furthermore, our observation that BMN scaling is recovered at large coupling in the hyperbolic Inozemtsev model is, while amusing, to be interpreted with great caution, as we are then potentially leaving the validity of the "asymptotic" Bethe ansatz (2.18). Indeed, when λ is large we should not really use the hyperbolic limit of the full elliptic model. Unfortunately it is not yet clear how to generalize the Bethe ansatz to this case.

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3. Stringing spins and spinning strings at two and three loops

3.1 Generalities

In the last section we showed that the Bethe ansatz for the Inozemtsev chain is indeed capable of reproducing the correct two- and three-loop anomalous dimensions of finite length two-impurity operators. Let us now turn to the "thermodynamic" situation of "long" operators with many impurities; i.e. we will study the operators of "length" $L = J_1 + J_2$ and "magnon-number" $M = J_2$

$$\mathrm{Tr}\, Z^{J_1} \Phi^{J_2} + \cdots , \tag{3.1}$$

in the limit where both J_1 and J_2 are large. At the one-loop level anomalous dimensions for these operators were obtained in [18] and it was established in [18, 19] that the result agreed with string theory predictions of Arutyunov, Frolov, Russo and Tseytlin [12, 16, 17], in two cases: The folded string (corresponding to the ground state) and the circular string (corresponding to an excited state). As we explained above, the Inozemtsev chain turns out to be intrinsically inconsistent with perturbative BMN scaling, starting at four loops. Just like BMN, the all-loop proposal of Frolov and Tseytlin also predicts an expansion in the BMN coupling $\lambda' = \lambda/J^2$. We immediately infer that the *perturbative* Inozemtsev ansatz therefore has no chance of reproducing these predictions beyond the three-loop level. On the other hand, up to three loops BMN scaling is possible, and, at the two-loop level, proven. As was shown in the previous sections, the three-loop dilatation operator of [27] can be emulated by the Inozemtsev chain. It is therefore clearly very interesting to work out the Inozemtsev predictions at two and three loops and compare to string theory. This will be done in the next sections, for both the folded and the circular string. At the two-loop level we can be sure that we are indeed computing the perturbative gauge theory result. This is in contradistinction to the three-loop case, where we need to assume the absence of the terms that break perturbative BMN scaling. Let us recall the string result for the two relevant cases of spinning folded and circular strings. It may be written in the following concise way. For the *folded* string we have the parametric expression

$$\left(\frac{\mathcal{J}_2}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2}\,, \tag{3.2}$$

$$\left(\frac{\mathcal{E}}{\mathrm{K}(t)}\right)^2 - \left(\frac{\mathcal{J}_1}{\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2}\,t\,, \tag{3.3}$$

while the *circular* string energy $\mathcal{E}$ is given by

$$\left(\frac{\mathcal{J}_1}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 - \left(\frac{\mathcal{J}_2}{(1-t)\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2 t}\,, \tag{3.4}$$

$$\left(\frac{\mathcal{E}}{\mathrm{K}(t)}\right)^2 - \left(\frac{t\mathcal{J}_1}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2 = \frac{4}{\pi^2}(1-t)\,, \tag{3.5}$$

where $\mathcal{E}, \mathcal{J}_1, \mathcal{J}_2$ are the string energies E and the angular momenta J_1,J_2 in units of the effective string tension λ (alias square root of the 't Hooft coupling), $E = \sqrt{\lambda}\ \mathcal{E}$, $J_{1,2} = \sqrt{\lambda}\ \mathcal{J}_{1,2}$. $\mathrm{K}(t)$ and $\mathrm{E}(t)$ are elliptic integrals of, respectively, the first and second kind. In both cases the upper equations (3.2), (3.4) determine the parameter t[9] as a function of J_1,J_2, while E is then obtained by substitution of t into, respectively, the lower equations (3.3), (3.5). These expressions may be expanded for large $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$; this results in an expansion of the energy as a Taylor series in the BMN coupling λ' through the identification (*cf.* eq. (1.1))

$$\lambda' = \frac{1}{\mathcal{J}^2} \, . \tag{3.6}$$

In order to obtain a finite result for the energy $\mathcal{E} = \mathcal{J} + \cdots$ we should introduce a rescaled string energy (=lowest charge) $\mathcal{Q}_2$:

$$\mathcal{Q}_2 = \frac{\mathcal{E}}{\mathcal{J}} \, . \tag{3.7}$$

One then has

$$\mathcal{Q}_2 = 1 + \sum_{k=1}^{\infty} \mathcal{Q}_2^{(k)} \, (\lambda')^k \, , \tag{3.8}$$

where the expansion coefficients $\mathcal{Q}_2^{(k)}$ only depend on the dimensionless ratio (termed "filling fraction" below)

$$\alpha \equiv \frac{\mathcal{J}_2}{\mathcal{J}} = \frac{J_2}{J} \, , \tag{3.9}$$

with $J = J_1 + J_2$. Since the expansion eq. (3.8) is in *integer* powers of λ', the basic proposal of Frolov and Tseytlin has been that the k-th expansion coefficient $\mathcal{Q}_2^{(k)}$ should be reproducible by a k-loop *perturbative* gauge theory calculation of the anomalous dimension of the operators eq. (3.1). This proposal generalizes the one of BMN, as the case of "J_2-impurity=J_2-magnon" BMN operators may be recovered from the folded string solution eqs. (3.2), (3.3) in the limit where J_2 stays finite (i.e. $\mathcal{J}_2 \to 0$), but the length of the operator still goes to infinity: $J \sim J_1 \to \infty$. In this case eq. (3.8) becomes

$$\mathcal{Q}_2 = 1 + \frac{J_2}{J}\left(\sqrt{1+\lambda'} - 1\right) \tag{3.10}$$

i.e. the $\mathcal{Q}_2^{(k)}$ become proportional to the Catalan numbers:

$$\mathcal{Q}_2^{(k)} = \frac{J_2}{J} 8 \left(-\frac{1}{4}\right)^{k+1} \frac{(2k-2)!}{k!(k-1)!} \, . \tag{3.11}$$

As discussed in the introduction, both the BMN and the Frolov-Tseytlin proposal make the highly non-trivial assumption that in the AdS/CFT system two different scaling procedures are nevertheless equivalent: Recall that in perturbative gauge theory the 't Hooft coupling λ is *small*, while the string calculation requires λ to be *large*. We are now ready to test the Frolov-Tseytlin proposal beyond the one-loop level, with the help of our long range Bethe ansatz eq. (2.18), (2.9).

[9] The parameter t is not to be confused with the Inozemtsev coupling constant $t = e^{-2\kappa}$ of the previous section.

3.2 Thermodynamic limit of the Inozemtsev-Bethe equations

The thermodynamic limit of the Inozemtsev-Bethe equations is obtained in close analogy to the procedure employed for the Heisenberg model, *cf.* [18]. One takes the logarithm of eq. (2.18) and assumes the length $L = J = J_1 + J_2$ to be large. This gives

$$J\ p_j = 2\pi n_j + 2\sum_{\substack{k=1\\k\neq j}}^{J_2} \frac{1}{\varphi_j - \varphi_k}\,. \tag{3.12}$$

In the case of the one-loop Bethe ansatz one works with the Bethe roots u_j, or, equivalently, the momenta p_j. In the present case one has, in addition to the $\{p_j\}$, a second set of discrete variables, namely the rapidities $\{\varphi_j\}$. The two sets are related through the coupling constant dependent relation eq. (2.9). In order to obtain equations which stay close in form to the original Bethe ansatz it is most convenient to use eqs. (3.12) to determine the distribution of the rapidities $\{\varphi_j\}$, instead of working with the momenta $\{p_j\}$. We can think of the rapidities as "deformed" Bethe roots, since we have $\varphi_j = u_j + \mathcal{O}(\lambda)$.

In the limit where the magnon number $M = J_2$ gets large, the rapidities are expected, after a rescaling, to accumulate along smooth contours in the complex φ plane, just as its undeformed leading order values u_j

$$\frac{\varphi_j}{J} \to \varphi \qquad \text{with } \rho(\varphi) = \frac{1}{J}\sum_{j=1}^{J_2} \delta\left(\varphi - \frac{\varphi_j}{J}\right), \tag{3.13}$$

where $\rho(\varphi)$ is a distribution density which is normalized to the filling fraction $\alpha = J_2/J$,

$$\int_{\mathcal{C}} d\varphi\, \rho(\varphi) = \alpha\,, \tag{3.14}$$

and $\mathcal{C}$ is the support of the density, i.e. the union of contours along which the rapidities are distributed. The Bethe equations (3.12) then turn into singular integral equations:

$$⨍_{\mathcal{C}} \frac{d\varphi'\, \rho(\varphi')\, \varphi}{\varphi' - \varphi} = \frac{1}{2}\, \varphi\, p(\varphi) + \pi\, n_{\mathcal{C}(\varphi)}\, \varphi\,. \tag{3.15}$$

$n_{\mathcal{C}(\varphi)}$ is the mode number at point φ. It should be constant along each contour. Here and in the following the slash through the integral sign implies a principal part prescription. This equation generalizes the continuum Bethe equations of [18]. On the right side the non-trivial potential $\varphi\, p(\varphi)/2$ appears, which is given by the scaling limit of the *inverse* of the rapidity $\varphi(p)\ p/2$ in eq. (2.9). Eq. (3.15) should be complemented by the momentum conservation condition $⨍_{\mathcal{C}} d\varphi\, \rho(\varphi)\, p(\varphi) = 0$ and the constraint $⨍_{\mathcal{C}} d\varphi\, \rho(\varphi)\, n_{\mathcal{C}(\varphi)} = 0$.

We would like to stress that the Inozemtsev-Bethe equation eq. (3.15) is expected to determine the spectrum of the long-range spin chain (and therefore, potentially, of the operators in eq. (3.1)) quite generally in the regime J_1, J_2 large (neglecting, however, the "wrapping" modes, as explained before). It is *not* necessarily restricted to perturbation theory. The information on the observables (namely, the commuting charges) of the long-range spin chain should then be contained in the *moments* $\bar{Q}_k$ of the rapidity distribution, in generalization of the result of [21, 20]:

$$\bar{Q}_k = \frac{1}{2} \int_{\mathcal{C}} d\varphi \, \frac{\rho(\varphi)}{\varphi^k} \,. \tag{3.16}$$

However, as far as determining the energy (=lowest charge) of the spin chain is concerned, finding the distribution of rapidities is not yet everything. We also need to uncover the relation between the "natural" observables $\bar{Q}_k$ and the observables we would like to measure, i.e. in particular, the eigenvalue of the dilatation operator. In the perturbative regime, up to three loops, this problem was solved in the last section. To this order, we are now ready to obtain the anomalous dimension of the operators eq. (3.1) in the two cases of interest (folded and circular).

3.3 Folded case

The folded case corresponds to the ground state of the operators eq. (3.1). The qualitative rapidity distribution is expected to be as in the one-loop case [18]. We have precisely two cuts symmetrically placed to both sides of the imaginary axis, with, respectively, mode numbers $\pm n$. Using the perturbative result of the last section, we find (putting the mode number n to one), up to three loops, the integral equation

$$⨍_a^b d\varphi' \, \frac{\rho(\varphi') \, \varphi^2}{\varphi'^2 - \varphi^2} = \frac{1}{4} - \frac{\pi}{2} \, \varphi + c_1 \, \frac{\lambda'}{\varphi^2} + c_2 \, \frac{\lambda'^2}{\varphi^4} \,, \tag{3.17}$$

which may be treated in a fashion identical to the one appearing in the multicritical $\mathrm{O}(n)$ matrix model [45]. The constants c_1, c_2 are found from the results of the previous section to be

$$c_1 = \frac{1}{32\pi^2} \qquad \text{and} \qquad c_2 = \frac{3}{512\pi^4} \,. \tag{3.18}$$

Let us define the moments $\bar{Q}_{2k}$ of the φ distribution by

$$\bar{Q}_{2k} = \int_a^b d\varphi \, \frac{\rho(\varphi)}{\varphi^{2k}} \,. \tag{3.19}$$

The energy Q_2 (i.e. the anomalous dimensions divided by J) is then determined[10] by

$$Q_2 = 1 - \frac{\lambda'}{4\pi^2} \, \bar{Q}_2 - e_1 \, \lambda'^2 \, \bar{Q}_4 - e_2 \, \lambda'^3 \, \bar{Q}_6 \,, \tag{3.20}$$

where the constants e_1, e_2 were found in the last section

$$e_1 = \frac{3}{64\pi^4} \qquad \text{and} \qquad e_2 = \frac{5}{512\pi^6} \,. \tag{3.21}$$

[10] Formally the moments $\bar{Q}_{2k}$, and in consequence also the energy Q_2, have an infinite regular expansion in the BMN coupling λ'. The leading order $\mathcal{O}(\lambda'^0)$ contribution to $\bar{Q}_{2k}$ is nothing but the $2k$'th one-loop commuting charge, *cf.* [21]. Obviously we should discard all terms higher then $\mathcal{O}(\lambda'^3)$ in Q_2 since the thermodynamic Inozemtsev-Bethe equation eq. (3.17) is accurately describing perturbative gauge theory only to at most $\mathcal{O}(\lambda'^3)$. As we discussed, in light of the results of [29], we can actually only be completely sure about this validity to $\mathcal{O}(\lambda'^2)$ and need to assume the vanishing of two so far undetermined constants to ensure BMN scaling and, in consequence, validity of eq. (3.17) to $\mathcal{O}(\lambda'^3)$.

It is easily verified that these equations are a generalization of the one-loop ones presented in [18]; their solution proceeds as before and we will thus be rather brief; for further details, see [21]. The normalization of the density is

$$\int_a^b d\varphi\ \rho(\varphi) = -\frac{\alpha}{2}\,. \tag{3.22}$$

with the by now standard trick [18, 19, 21] of analytic continuation to (formally) negative J, resulting in a negative filling fraction $\alpha = J_2/J < 0$ (this flips the complex cuts onto the real axis). It is again useful to introduce a generating function for the moments $\bar{Q}_{2k}$:

$$H(\varphi) = -\frac{\alpha}{2} + \sum_{k=1}^{\infty} \bar{Q}_{2k}\ \varphi^{2k}\,, \qquad \text{i.e.} \qquad H(\varphi) = \int_a^b d\varphi'\ \rho(\varphi')\ \frac{\varphi'^2}{\varphi'^2 - \varphi^2}\,. \tag{3.23}$$

We find the following generalization of the result eq.(2.24) in [21]:

$$\begin{aligned} H(\varphi) = &-\frac{\alpha}{2} + \frac{1}{4} + c_1\ \frac{\lambda'}{\varphi^2} + c_2\ \frac{\lambda'^2}{\varphi^4} - \frac{a^2}{b}\sqrt{\frac{b^2-\varphi^2}{a^2-\varphi^2}}\ \Pi\left(-q\frac{\varphi^2}{a^2-\varphi^2}, q\right) - \\ &-\frac{1}{ab}\sqrt{(b^2-\varphi^2)(a^2-\varphi^2)}\left[c_1\ \lambda'\ \frac{1}{\varphi^2} + c_2\ \lambda'^2\left(\frac{1}{\varphi^4} + \frac{a^2+b^2}{2a^2b^2}\frac{1}{\varphi^2}\right)\right], \end{aligned} \tag{3.24}$$

where the modulus q is, as before, $q = 1 - a^2/b^2$. The Π function is the elliptic integral of the third kind; for our conventions we refer to [21]. One then finds that the left edge a of the cut is determined through the equation

$$\frac{1}{4} - \mathrm{K}(q)\ a + c_1\ \lambda'\left(1 - \frac{q}{2}\right)\frac{1}{a^2} + c_2\lambda'^2\left(1 - q + \frac{3}{8}q^2\right)\frac{1}{a^4} = 0\,, \tag{3.25}$$

while the right edge b is, of course, $b = a/\sqrt{1-q}$. Finally, one has an equation relating the filling fraction to the modulus q:

$$\alpha = \frac{1}{2} - \frac{2a\mathrm{E}(q)}{\sqrt{1-q}} + c_1\ \lambda'\ \frac{2\sqrt{1-q}}{a^2} + c_2\ \lambda'^2\ \frac{(2-q)\sqrt{1-q}}{a^4}\,. \tag{3.26}$$

$\mathrm{K}(q)$ and $\mathrm{E}(q)$ in eqs. (3.25), (3.26) are elliptic integrals of the, respectively, first and second kind. We are now able to obtain the gauge theory result up to three loops, using eq. (3.20).This is done by expanding both the modulus $q = q_0 + q_1\lambda' + q_2\lambda'^2$ and the interval boundary $a = a_0 + a_1\lambda' + a_2\lambda'^2$ in powers of λ', and eliminating with the help of eqs. (3.25), (3.26) all constants except for the leading order modulus q_0, which, as before, conveniently parametrizes the filling fraction α through

$$-\alpha = -\frac{J_2}{J} = \frac{1}{2\sqrt{1-q_0}}\frac{\mathrm{E}(q_0)}{\mathrm{K}(q_0)} - \frac{1}{2}\,. \tag{3.27}$$

This gives the final, three-loop result

$$
\begin{aligned}
Q_2 = {} & 1+\frac{1}{2\pi^2}\mathrm{K}(q_0)(2\mathrm{E}(q_0)-(2-q_0)\mathrm{K}(q_0))\lambda' + \\
& +\frac{1}{8\pi^4}\mathrm{K}(q_0)^3\left(4(2-q_0)\mathrm{E}(q_0)-(8-8q_0+q_0^2)\mathrm{K}(q_0)\right)\lambda'^2 + \\
& +\frac{1}{4\pi^6}\frac{\mathrm{K}(q_0)^5}{(\mathrm{E}(q_0)-\mathrm{K}(q_0))(\mathrm{E}(q_0)-(1-q_0)\mathrm{K}(q_0))}\times \\
& \times\Big((8-8q_0+3q_0^2)\mathrm{E}(q_0)^3-(2-q_0)(12-12q_0+q_0^2)\mathrm{E}(q_0)^2\mathrm{K}(q_0) + \\
& \quad +3(1-q_0)(8-8q_0+q_0^2)\mathrm{E}(q_0)\mathrm{K}(q_0)^2-4(1-q_0)^2(2-q_0)\mathrm{K}(q_0)^3\Big)\,\lambda'^3\,. \qquad (3.28)
\end{aligned}
$$

We notice that the two-loop ($\mathcal{O}(\lambda'^2)$ contribution in eq. (3.28) very closely resembles the eigenvalue of the fourth higher charge Q_4, *cf.* eq.(2.25) in [21]. This finds a natural explanation in terms of the structure of the energy expression eq. (3.20): Indeed the two-loop energy is "mostly" given by the contribution from the fourth conserved charge of the Heisenberg XXX chain, with a "small" correction, namely the λ' correction to the second Heisenberg charge.

Let us now compare this to the string prediction eqs. (3.2), (3.3). The latter is conveniently rewritten in Lagrange-inversion form, using eqs. (3.6), (3.9):

$$
\begin{aligned}
\lambda' &= \frac{\pi^2}{4\mathrm{K}(t_0)^2}\left[\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2-\left(\frac{\mathrm{E}(t_0)}{\mathrm{E}(t)}\right)^2\right], \\
\mathcal{Q}_2 &= \frac{\mathrm{K}(t)}{\mathrm{K}(t_0)}\sqrt{(1-t)\left(\frac{\mathrm{E}(t_0)}{\mathrm{E}(t)}\right)^2+t\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2}\,. \qquad (3.29)
\end{aligned}
$$

where the "string modulus" t_0 is found to parametrize the filling fraction through

$$
\alpha = 1-\frac{\mathrm{E}(t_0)}{\mathrm{K}(t_0)}\,. \qquad (3.30)
$$

One now expands both equations in t around the point t_0. The upper equation gives λ' as a power series in the variable $(t-t_0)$. Power series inversion yields t as a power series in λ'. Finally, substitution of this series into the expanded lower equation gives $\mathcal{Q}_2$ as a series in λ', i.e. the sought coefficients in eq. (3.8). We then find, up to third order in λ',

$$
\begin{aligned}
\mathcal{Q}_2 = {} & 1+\frac{2}{\pi^2}\mathrm{K}(t_0)\left[\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)\right]\lambda' + \\
& +\frac{2}{\pi^4}\mathrm{K}(t_0)^3\left[(1-2t_0)\mathrm{E}(t_0)-(1-t_0)^2\mathrm{K}(t_0)\right]\lambda'^2 + \\
& +\frac{4}{\pi^6}\frac{\mathrm{K}(t_0)^5}{\mathrm{E}(t_0)^2-2(1-t_0)\mathrm{E}(t_0)\mathrm{K}(t_0)+(1-t_0)\mathrm{K}(t_0)^2}\times \\
& \times\Big((1-7t_0+7t_0^2)\mathrm{E}(t_0)^3-(1-t_0)(3-14t_0+7t_0^2)\mathrm{E}(t_0)^2\mathrm{K}(t_0) + \qquad (3.31) \\
& \quad +(1-t_0)(3-2t_0)(1-3t_0+t_0^2)\mathrm{E}(t_0)\mathrm{K}(t_0)^2-(1-t_0)^4\mathrm{K}(t_0)^3\Big)\lambda'^3+\mathcal{O}(\lambda'^4)\,,
\end{aligned}
$$

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In order to compare to gauge theory, we need to parametrize both results in the same way, *cf.* eqs. (3.27), (3.30). In fact, we can apply the Gauss-Landen transformation of [19], which relates the string and gauge theory modulus through

$$t_0 = -\frac{(1-\sqrt{1-q_0})^2}{4\sqrt{1-q_0}}\,, \tag{3.32}$$

to rewrite the three-loop string energy $\mathcal{Q}_2$ in terms of the gauge modulus q_0:

$$\begin{aligned}
\mathcal{Q}_2 = {}& 1 + \frac{1}{2\pi^2}\mathrm{K}(q_0)(2\mathrm{E}(q_0) - (2-q_0)\mathrm{K}(q_0))\lambda' + \qquad (3.33)\\
&+\frac{1}{8\pi^4}\mathrm{K}(q_0)^3\left(4(2-q_0)\mathrm{E}(q_0) - (8-8q_0+q_0^2)\mathrm{K}(q_0)\right)\lambda'^2 + \\
&+\frac{1}{16\pi^6}\frac{\mathrm{K}(q_0)^5}{(\mathrm{E}(q_0)-\mathrm{K}(q_0))(\mathrm{E}(q_0)-(1-q_0)\mathrm{K}(q_0))}\times \\
&\times\Big(2(16-16q_0+7q_0^2)\mathrm{E}(q_0)^3 - (2-q_0)(48-48q_0+7q_0^2)\mathrm{E}(q_0)^2\mathrm{K}(q_0) + \\
&\quad + (96-192q_0+114q_0^2-18q_0^3+q_0^4)\mathrm{E}(q_0)\mathrm{K}(q_0)^2 - \\
&\quad - (1-q_0)(2-q_0)(16-16q_0+q_0^2)\mathrm{K}(q_0)^3\Big)\,\lambda'^3 + \mathcal{O}(\lambda'^4)\,. \qquad (3.34)
\end{aligned}$$

Comparing the string theory result eq. (3.33) to the gauge theory result eq. (3.28), we see that the $\mathcal{O}(\lambda'^2)$ two-loop terms precisely agree! Our joy about this amazing result would be even greater if this was also the case for the $\mathcal{O}(\lambda'^3)$ three-loop contribution. However, here we notice, despite a close structural resemblance (in particular, the denominators of the rather involved formulas agree), that the detailed form of the gauge and string theory results are *different*. Furthermore, these analytic findings explain the numerical results of [19] for the ground state gauge energies of the folded case at half filling $\alpha = \frac{1}{2}$. There the thermodynamic ground state energies of the gauge operators were estimated by extrapolation of the numerically exact diagonalization of the dilatation operator for states up to $J = 16$. The method showed excellent gauge-string theory agreement (to about 1%) at one and two-loops, but a curious 17% deviation at three loops. Having at our hands the exact Inozemtsev-Bethe results eq. (3.28) we can find[11] the numerically exact values of the three-loop ground state energies at half-filling:[12]

$$Q_2 = 1 + 0.356016\lambda' - 0.212347\lambda'^2 + 0.212147\lambda'^3\,, \tag{3.35}$$

while the string result eqs. (3.31), (3.33) gives

$$\mathcal{Q}_2 = 1 + 0.356016\lambda' - 0.212347\lambda'^2 + 0.181673\lambda'^3\,, \tag{3.36}$$

We see that this *explains* the numerical results in table 1 of [19]. It shows that the 17% deviation noticed there is *not* due to numerical inaccuracy of the extrapolation method — weak coupling gauge and strong coupling string theory results really begin to differ at three loops.

[11]This nice test of the validity of our method was first suggested to us by N. Beisert.

[12]It is technically convenient to use the inverse Gauss-Landen transform on the gauge result eq. (3.28); $\alpha = \frac{1}{2}$ corresponds to the modulus $t_0 = 0.8261148$, where $2\mathrm{E}(t_0) = \mathrm{K}(t_0)$.

The attentive reader might wonder whether the three-loop disagreement might disappear when the above constants eqs. (3.18), (3.21) are chosen in a different way: Leaving aside the Inozemtsev chain, the general form eqs. (3.17), (3.20) of "perturbing" the Bethe equations of the one-loop Heisenberg model is of course very natural. However, this is not the case. One checks (1) that requiring agreement between gauge and string theory at two-loops fixes c_1, e_1 to precisely the values derived from the Inozemtsev-Bethe ansatz, that furthermore (2) *no* choice of c_2, e_2 enables one to match the solution of the deformed Bethe equations and the string prediction, and that finally (3) the three-loop gauge result eq. (3.35) is very sensitive to the precise values of c_2, e_2 and the beautiful agreement with table 1 of [19], *cf.* eqs(3.35), (3.36) is only recovered if we chose the "Inozemtsev values", i.e. eqs. (3.18), (3.21).

One might wonder whether the Inozemtsev chain reproduces the string result in the non-perturbative BMN regime where $\lambda \to \infty$ in conjunction with $J \to \infty$, while keeping $\lambda' = \lambda/J^2$ small. Unfortunately, in this limit we cannot really trust our asymptotic Bethe ansatz, which requires the length of the spin chain to be larger than the considered order or perturbation theory. This is because at strong coupling high orders in λ become relevant, and might, potentially, lead to further corrections in the parameter λ'. In the gauge theory these diagrams come from "wrapping" Feynman diagrams, which involve the interaction of all components inside the trace of the operator. Analogously, in the Inozemtsev model we would have to investigate the strong coupling thermodynamics of the full elliptic (as opposed to the hyperbolic) model. The appropriate Bethe ansatz is however not yet known (if it exists at all!), and it might well lead to a modification of the scaled Bethe equations. One may nevertheless wonder whether the currently treatable hyperbolic case might reconcile, by itself, the hyberbolic Inozemtsev model and string theory. Indeed, as we discussed in section 2.2, in the former model a finite scaled *strong coupling* rapidity exists, *cf.* eq. (2.13). Upon inversion we find

$$\frac{1}{4}\,\varphi\,p(\varphi) = \frac{\pi\,\varphi}{4\sqrt{\frac{3}{2}\lambda'}}\log\frac{1+\sqrt{\frac{3}{2}\lambda'}\frac{1}{2\pi\varphi}}{1-\sqrt{\frac{3}{2}\lambda'}\frac{1}{2\pi\varphi}}\,. \tag{3.37}$$

However, it is now easily seen that this also does not cure the disagreement, due to the just mentioned finding (2) (no choice of c_2, e_2 enables one to match the three-loop string and spin chain results). This is because expanding eq.(59) in λ' yields at three loops a potential of the same form as found on the right side of eq. (3.17). We conclude that the hyperbolic (i.e. wrappings are neglected) Inozemtsev chain at strong coupling also fails to reproduce the string prediction. One nevertheless checks the quite interesting and non-trivial fact that at two loops the strongly coupled hyperbolic chain again reproduces the same potential as the weakly coupled chain.

To summarize: The hyperbolic Inozemtsev chain does not reproduce three-loop semi-classical string theory at either weak or strong coupling. This means, lest AdS/CFT is wrong, or the string derivation is fraught with a currently unknown mistake, that we need to extend the Bethe ansatz to the non-asymptotic situation. Note that this latter conclusion is *independent* of the question whether the Inozemtsev model itself needs to be modified at

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four loops and beyond in order to describe $\mathcal{N}=4$ theory. This is because the disagreement occurs at *three*, not *four* loops, where we have proven the applicability of our model to gauge theory in the last section.

For further confirmation of these findings, let us now turn to the case of the circular string.

3.4 Circular case

Here we will be rather brief, as the details are easily filled in by combining the results and notations of [18, 19, 21] and the methodology of the last section. The Inozemtsev-Bethe equation reads in this case

$$\fint_c^d d\varphi'\,\tilde{\sigma}(\varphi')\,\frac{\varphi^2}{\varphi^2-\varphi'^2}=\frac{1}{4}-\varphi\log\frac{\varphi+c}{\varphi-c}-c_1\frac{\lambda'}{\varphi^2}+c_2\,\frac{\lambda'^2}{\varphi^4}\,, \tag{3.38}$$

where we have rotated the variable $\varphi\to i\varphi$, while the three-loop gauge energy Q_2 is found from[13]

$$Q_2=1+\frac{\lambda'}{4\pi^2}\,\bar{Q}_2-e_1\,\lambda'^2\,\bar{Q}_4+e_2\,\lambda'^3\,\bar{Q}_6\,, \tag{3.39}$$

with the constants c_1,c_2,e_1,e_2 again given by eqs. (3.18), (3.21). The $\bar{Q}_{2k}$ are the moments of the distribution density $\tilde{\sigma}(\varphi)$ and obtained by

$$\bar{Q}_{2k}=\frac{2}{2k-1}\frac{1}{c^{2k-1}}-\int_c^d d\varphi\frac{\tilde{\sigma}(\varphi)}{\varphi^{2k}}\,. \tag{3.40}$$

They are generated by the resolvent

$$H(\varphi)=\frac{\alpha}{2}-\sum_{k=1}^{\infty}\bar{Q}_{2k}\,\varphi^{2k}\qquad\text{i.e.}\qquad H(\varphi)=2c+\varphi\log\frac{c-\varphi}{c+\varphi}+\int_c^d d\varphi'\tilde{\sigma}(\varphi')\frac{\varphi'^2}{\varphi'^2-\varphi^2}\,. \tag{3.41}$$

The normalization is

$$2c+\int_c^d d\varphi\;\tilde{\sigma}(x)=\frac{\alpha}{2}\,, \tag{3.42}$$

where the filling fraction is $\alpha=J_2/J>0$. The solution involves again the elliptic integral of the third kind, Π, with modulus $r=c^2/d^2$ and reads

$$\begin{aligned}H(\varphi)\;=\;&\frac{\alpha}{2}-\frac{1}{4}+c_1\,\frac{\lambda'}{\varphi^2}-c_2\,\frac{\lambda'^2}{\varphi^4}+\frac{2}{d}\sqrt{(d^2-\varphi^2)(c^2-\varphi^2)}\;\Pi\left(\frac{\varphi^2}{d^2},r\right)-\\&-\frac{1}{cd}\sqrt{(d^2-\varphi^2)(c^2-\varphi^2)}\;\left[c_1\,\lambda'\,\frac{1}{\varphi^2}-c_2\,\lambda'^2\left(\frac{1}{\varphi^4}+\frac{c^2+d^2}{2c^2d^2}\frac{1}{\varphi^2}\right)\right],\end{aligned} \tag{3.43}$$

while the conditions fixing the interval boundaries c, $d=c/\sqrt{r}$ and the modulus r are

$$-\frac{1}{4}+2\mathrm{K}(r)\;c+c_1\,\lambda'\,\frac{1+r}{2}\,\frac{1}{c^2}-c_2\,\lambda'^2\,\frac{3+2r+3r^2}{8}\,\frac{1}{c^4}=0\,, \tag{3.44}$$

[13] The sign changes in eqs. (3.38), (3.39) w.r.t. their folded analogs eqs. (3.17), (3.20) are due to the rotation $\varphi\to i\varphi$, and a slightly different definition of the moments in both cases. See also [21].

and

$$\alpha = \frac{4c}{\sqrt{r}}\left[\left(\sqrt{r}-1\right)\mathrm{K}(r)+\mathrm{E}(r)\right]+c_1\ \lambda'\ \frac{(1-\sqrt{r})^2}{c^2}-c_2\ \lambda'^2\ \frac{(1-\sqrt{r})^2(3+2\sqrt{r}+3r)}{4c^4}\,. \tag{3.45}$$

Expanding the modulus $r = r_0 + r_1\lambda' + r_2\lambda'^2$ and the interval boundary c, and eliminating all constants except r_0, we find the relation

$$\alpha = \frac{1}{2} - \frac{1}{2\sqrt{r_0}} + \frac{1}{2\sqrt{r_0}}\frac{\mathrm{E}(r_0)}{\mathrm{K}(r_0)}\,. \tag{3.46}$$

The three-loop gauge energy is then

$$\begin{aligned} Q_2 &= 1+\frac{2}{\pi^2}\mathrm{K}(r_0)\left(2\mathrm{E}(r_0)-(1-r_0)\mathrm{K}(r_0)\right)\ \lambda' - \\ &-\frac{2}{\pi^4}\mathrm{K}(r_0)^3\left(4(1+r_0)\mathrm{E}(r_0)-(1-r_0)(3+r_0)\mathrm{K}(r_0)\right)\ \lambda'^2 + \\ &+\frac{16}{\pi^6}\frac{\mathrm{K}(r_0)^5}{\mathrm{E}(r_0)\left(\mathrm{E}(r_0)-(1-r_0)\mathrm{K}(r_0)\right)}\times \\ &\times\Big((3+2r_0+3r_0^2)\mathrm{E}(r_0)^3-(1-r_0)(8+3r_0+r_0^2)\mathrm{E}(r_0)^2\mathrm{K}(r_0)+ \\ &\quad+(1-r_0)^2(7+r_0)\mathrm{E}(r_0)\mathrm{K}(r_0)^2-2(1-r_0)^3\mathrm{K}(r_0)^3\Big)\lambda'^3\,. \end{aligned} \tag{3.47}$$

Now we will again compare this to the string prediction, eqs. (3.4), (3.5). Written in Lagrange-inversion form, we have

$$\begin{aligned} \lambda' &= \frac{\pi^2 t}{4t_0^2\mathrm{K}(t_0)^2}\left[\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2-\left(\frac{\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)}{\mathrm{E}(t)-(1-t)\mathrm{K}(t)}\right)^2\right], \\ \mathcal{Q}_2 &= \frac{t\mathrm{K}(t)}{t_0\mathrm{K}(t_0)}\sqrt{\frac{1}{t}\left(\frac{\mathrm{K}(t_0)-\mathrm{E}(t_0)}{\mathrm{K}(t)-\mathrm{E}(t)}\right)^2-\frac{1-t}{t}\left(\frac{\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)}{\mathrm{E}(t)-(1-t)\mathrm{K}(t)}\right)^2}\,, \end{aligned} \tag{3.48}$$

where now the string modulus t_0 is parametrizing the filling fraction through

$$\alpha = 1 - \frac{1}{t_0} + \frac{1}{t_0}\,\frac{\mathrm{E}(t_0)}{\mathrm{K}(t_0)}. \tag{3.49}$$

As before, power series inversion and substitution gives, from eqs. (3.48),

$$\begin{aligned} \mathcal{Q}_2 &= 1+\frac{2}{\pi^2}\mathrm{E}(t_0)\mathrm{K}(t_0)\ \lambda' - \frac{2}{\pi^4}\mathrm{K}(t_0)^3\left((2-t_0)\mathrm{E}(t_0)-(1-t_0)\mathrm{K}(t_0)\right)\ \lambda'^2 + \\ &+\frac{4}{\pi^6}\frac{\mathrm{K}(t_0)^5}{\mathrm{E}(t_0)^2-(1-t_0)\mathrm{K}(t_0)^2}\times \\ &\times\Big((7-7t_0+t_0^2)\mathrm{E}(t_0)^3-7(2-t_0)(1-t_0)\mathrm{E}(t_0)^2\mathrm{K}(t_0)+ \\ &\quad+(1-t_0)(9-9t_0+t_0^2)\mathrm{E}(t_0)\mathrm{K}(t_0)^2-(2-t_0)(1-t_0)^2\mathrm{K}(t_0)^3\Big)\lambda'^3+\mathcal{O}(\lambda'^4)\,. \end{aligned} \tag{3.50}$$

After the Gauss-Landen transformation

$$t_0 = -\frac{4\sqrt{r_0}}{(1-\sqrt{r_0})^2} \tag{3.51}$$

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the string energy up to three-loop order becomes

$$
\begin{aligned}
\mathcal{Q}_2 = & \; 1 + \frac{2}{\pi^2}\mathrm{K}(r_0)\left(2\mathrm{E}(r_0) - (1-r_0)\mathrm{K}(r_0)\right)\lambda' - \\
& - \frac{2}{\pi^4}\mathrm{K}(r_0)^3\left(4(1+r_0)\mathrm{E}(r_0) - (1-r_0)(3+r_0)\mathrm{K}(r_0)\right)\lambda'^2 \\
& + \frac{4}{\pi^6}\frac{\mathrm{K}(r_0)^5}{\mathrm{E}(r_0)\left(\mathrm{E}(r_0)-(1-r_0)\mathrm{K}(r_0)\right)} \times \\
& \times\Big(2(7+2r_0+7r_0^2)\mathrm{E}(r_0)^3 - (1-r_0)(35+6r_0+7r_0^2)\mathrm{E}(r_0)^2\mathrm{K}(r_0) \\
& \quad + (1-r_0)^2(29+2r_0+r_0^2)\mathrm{E}(r_0)\mathrm{K}(r_0)^2 - 8(1-r_0)^3\mathrm{K}(r_0)^3\Big)\lambda'^3 + \mathcal{O}(\lambda'^4)\,. \qquad (3.52)
\end{aligned}
$$

The result of the comparison between the string theory prediction eq. (3.52) and the gauge theory computation eq. (3.47) leads to the same conclusion as the previously treated case of the folded string. There is a beautiful and non-trivial agreement at the two-loop order; on the other hand, the three-loop terms, despite great structural similarity, are definitely different. Again, varying the constants c_2, e_2 in eqs. (3.38), (3.39) does not help.

It is instructive to consider the special case of half-filling $\alpha = 1/2$. Here the solution becomes algebraic, and the string result reads, to all orders

$$\mathcal{Q}_2 = \sqrt{1+\lambda'} = 1 + \frac{\lambda'}{2} - \frac{\lambda'^2}{8} + \frac{\lambda'^3}{16} + \cdots\,. \qquad (3.53)$$

The three-loop gauge theory result, on the other hand, under the assumption of the validity of three-loop perturbative BMN scaling, yields

$$Q_2 = 1 + \frac{\lambda'}{2} - \frac{\lambda'^2}{8} + 0\lambda'^3\,. \qquad (3.54)$$

The three-loop contribution to the "circular" state *vanishes*!

3.5 A curious observation

It is clearly of some interest to pin down the difference between three-loop perturbation theory, and the string prediction. For the folded case we have

$$Q_2 - \mathcal{Q}_2 = -\frac{q_0^2}{16\pi^6}\mathrm{K}(q_0)^5\left(2\mathrm{E}(q_0) - (2-q_0)\mathrm{K}(q_0)\right)\lambda'^3\,, \qquad (3.55)$$

and in the circular case

$$Q_2 - \mathcal{Q}_2 = -\frac{4(1-r_0)^2}{\pi^6}\mathrm{K}(r_0)^5\left(2\mathrm{E}(r_0) - (1-r_0)\mathrm{K}(q_0)\right)\lambda'^3\,. \qquad (3.56)$$

Curiously, this looks like a non-linear (and therefore, from the point of view of the perturbative spin chain, non-local) contribution to the energy. In *both* cases the difference is proportional to the product of the one-loop second moment $\bar{Q}_2^{(1)}$ and its leading correction $\bar{Q}_2^{(2)}$:

$$Q_2 - \mathcal{Q}_2 = -\frac{\lambda'^3}{32\pi^4}\,\bar{Q}_2^{(1)}\,\bar{Q}_2^{(2)} \qquad \text{where} \qquad \bar{Q}_2 = \bar{Q}_2^{(1)} + \lambda'\bar{Q}_2^{(2)} + \mathcal{O}(\lambda'^2) \qquad (3.57)$$

But the significance or wider validity of this observation remains unclear to us at the moment. It would be interesting to see whether the same "recipe" also allows to account for the three-loop near-plane wave discrepancy of Callan *et al.* [28].

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References

[1] D. Berenstein, J.M. Maldacena and H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **04** (2002) 013 [hep-th/0202021].

[2] J.M. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200];
E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150];
S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] R.R. Metsaev, *Type iib Green-Schwarz superstring in plane wave Ramond-Ramond background*, *Nucl. Phys.* **B 625** (2002) 70 [hep-th/0112044];
R.R. Metsaev and A.A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background*, *Phys. Rev.* **D 65** (2002) 126004 [hep-th/0202109].

[4] A. Pankiewicz, *Strings in plane wave backgrounds*, *Fortschr. Phys.* **51** (2003) 1139 [hep-th/0307027];
J.C. Plefka, *Lectures on the plane-wave string/gauge theory duality*, *Fortschr. Phys.* **52** (2004) 264 [hep-th/0307101];
C. Kristjansen, *Quantum mechanics, random matrices and BMN gauge theory*, *Acta Phys. Polon.* **B34** (2003) 4949 [hep-th/0307204];
D. Sadri and M.M. Sheikh-Jabbari, *The plane-wave/super Yang-Mills duality*, hep-th/0310119.

[5] C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *A new double-scaling limit of $N = 4$ super Yang-Mills theory and pp-wave strings*, *Nucl. Phys.* **B 643** (2002) 3 [hep-th/0205033];
N.R. Constable et al., *Pp-wave string interactions from perturbative Yang-Mills theory*, *J. High Energy Phys.* **07** (2002) 017 [hep-th/0205089];
N. Beisert, C. Kristjansen, J. Plefka, G.W. Semenoff and M. Staudacher, *Bmn correlators and operator mixing in $N = 4$ super yang- mills theory*, *Nucl. Phys.* **B 650** (2003) 125 [hep-th/0208178];
N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, *Operator mixing and the bmn correspondence*, *J. High Energy Phys.* **10** (2002) 068 [hep-th/0209002];
N. Beisert, C. Kristjansen, J. Plefka and M. Staudacher, *Bmn gauge theory as a quantum mechanical system*, *Phys. Lett.* **B 558** (2003) 229 [hep-th/0212269].

[6] J.A. Minahan and K. Zarembo, *The bethe-ansatz for $N = 4$ super Yang-Mills*, *J. High Energy Phys.* **03** (2003) 013 [hep-th/0212208].

JHEP06(2004)001

[7] L.N. Lipatov, *High-energy asymptotics of multicolor QCD and exactly solvable lattice models*, *Sov. Phys. JETP Lett.* **59** (1994) 596 [hep-th/9311037];
L.D. Faddeev and G.P. Korchemsky, *High-energy QCD as a completely integrable model*, *Phys. Lett.* **B 342** (1995) 311 [hep-th/9404173];
V.M. Braun, S.E. Derkachov and A.N. Manashov, *Integrability of three-particle evolution equations in QCD*, *Phys. Rev. Lett.* **81** (1998) 2020 [hep-ph/9805225];
A.V. Belitsky, *Fine structure of spectrum of twist-three operators in QCD*, *Phys. Lett.* **B 453** (1999) 59 [hep-ph/9902361];
V.M. Braun, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Baryon distribution amplitudes in QCD*, *Nucl. Phys.* **B 553** (1999) 355 [hep-ph/9902375];
A.V. Belitsky, *Integrability and wkb solution of twist-three evolution equations*, *Nucl. Phys.* **B 558** (1999) 259 [hep-ph/9903512]; *Renormalization of twist-three operators and integrable lattice models*, *Nucl. Phys.* **B 574** (2000) 407 [hep-ph/9907420];
S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Evolution equations for quark gluon distributions in multi- color QCD and open spin chains*, *Nucl. Phys.* **B 566** (2000) 203 [hep-ph/9909539].

[8] N. Beisert, *The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 676** (2004) 3 [hep-th/0307015].

[9] N. Beisert and M. Staudacher, *The $N = 4$ sym integrable super spin chain*, *Nucl. Phys.* **B 670** (2003) 439 [hep-th/0307042].

[10] G. Mandal, N.V. Suryanarayana and S.R. Wadia, *Aspects of semiclassical strings in AdS_5*, *Phys. Lett.* **B 543** (2002) 81 [hep-th/0206103];
I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev.* **D 69** (2004) 046002 [hep-th/0305116];
B.C. Vallilo, *Flat currents in the classical $AdS_5 \times S^5$ pure spinor superstring*, *J. High Energy Phys.* **03** (2004) 037 [hep-th/0307018];
L.F. Alday, *Nonlocal charges on $AdS_5 \times S^5$ and pp-waves*, *J. High Energy Phys.* **12** (2003) 033 [hep-th/0310146].

[11] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys.* **B 636** (2002) 99 [hep-th/0204051].

[12] S. Frolov and A.A. Tseytlin, *Multi-spin string solutions in $AdS_5 \times S^5$*, *Nucl. Phys.* **B 668** (2003) 77 [hep-th/0304255].

[13] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 007 [hep-th/0204226].

[14] H.J. de Vega, A.L. Larsen and N. Sanchez, *Semiclassical quantization of circular strings in de Sitter and anti-de Sitter space-times*, *Phys. Rev.* **D 51** (1995) 6917 [hep-th/9410219].

[15] J.G. Russo, *Anomalous dimensions in gauge theories from rotating strings in $AdS_5 \times S^5$*, *J. High Energy Phys.* **06** (2002) 038 [hep-th/0205244];
J.A. Minahan, *Circular semiclassical string solutions on $AdS_5 \times S^5$*, *Nucl. Phys.* **B 648** (2003) 203 [hep-th/0209047];
A.A. Tseytlin, *Semiclassical quantization of superstrings: $AdS_5 \times S^5$ and beyond*, *Int. J. Mod. Phys.* **A 18** (2003) 981 [hep-th/0209116];
S. Frolov and A.A. Tseytlin, *Quantizing three-spin string solution in $AdS_5 \times S^5$*, *J. High Energy Phys.* **07** (2003) 016 [hep-th/0306130].

[16] S. Frolov and A.A. Tseytlin, *Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors*, *Phys. Lett.* **B 570** (2003) 96 [hep-th/0306143].

[17] G. Arutyunov, S. Frolov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$ and integrable systems*, *Nucl. Phys.* **B 671** (2003) 3 [hep-th/0307191];
G. Arutyunov, J. Russo and A.A. Tseytlin, *Spinning strings in $AdS_5 \times S^5$: new integrable system relations*, *Phys. Rev.* **D 69** (2004) 086009 [hep-th/0311004].

[18] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing spins and spinning strings*, *J. High Energy Phys.* **09** (2003) 010 [hep-th/0306139].

[19] N. Beisert, S. Frolov, M. Staudacher and A.A. Tseytlin, *Precision spectroscopy of AdS/CFT*, *J. High Energy Phys.* **10** (2003) 037 [hep-th/0308117].

[20] J. Engquist, J.A. Minahan and K. Zarembo, *Yang-Mills duals for semiclassical strings on $AdS_5 \times S^5$*, *J. High Energy Phys.* **11** (2003) 063 [hep-th/0310188].

[21] G. Arutyunov and M. Staudacher, *Matching higher conserved charges for strings and spins*, *J. High Energy Phys.* **03** (2004) 004 [hep-th/0310182].

[22] A.A. Tseytlin, *Spinning strings and AdS/CFT duality*, hep-th/0311139.

[23] M. Bianchi, M.B. Green, S. Kovacs and G. Rossi, *Instantons in supersymmetric Yang-Mills and d-instantons in IIB superstring theory*, *J. High Energy Phys.* **08** (1998) 013 [hep-th/9807033];
G. Arutyunov and E. Sokatchev, *On a large-N degeneracy in $N = 4$ SYM and the AdS/CFT correspondence*, *Nucl. Phys.* **B 663** (2003) 163 [hep-th/0301058].

[24] D.J. Gross, A. Mikhailov and R. Roiban, *Operators with large R charge in $N = 4$ Yang-Mills theory*, *Ann. Phys. (NY)* **301** (2002) 31 [hep-th/0205066].

[25] B. Eden, *On two fermion bmn operators*, *Nucl. Phys.* **B 681** (2004) 195 [hep-th/0307081].

[26] A. Santambrogio and D. Zanon, *Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*, *Phys. Lett.* **B 545** (2002) 425 [hep-th/0206079].

[27] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys.* **B 664** (2003) 131 [hep-th/0303060].

[28] C.G. Callan Jr. et al., *Quantizing string theory in $AdS_5 \times S^5$: beyond the pp-wave*, *Nucl. Phys.* **B 673** (2003) 3 [hep-th/0307032].

[29] N. Beisert, *The su(2—3) dynamic spin chain*, *Nucl. Phys.* **B 682** (2004) 487 [hep-th/0310252].

[30] N. Beisert, *BMN operators and superconformal symmetry*, *Nucl. Phys.* **B 659** (2003) 79 [hep-th/0211032].

[31] N.-w. Kim and J. Plefka, *On the spectrum of pp-wave matrix theory*, *Nucl. Phys.* **B 643** (2002) 31 [hep-th/0207034];
N.-w. Kim, T. Klose and J. Plefka, *Plane-wave matrix theory from $N = 4$ super Yang-Mills on $R \times S^3$*, *Nucl. Phys.* **B 671** (2003) 359 [hep-th/0306054];
T. Klose and J. Plefka, *On the integrability of large-N plane-wave matrix theory*, *Nucl. Phys.* **B 679** (2004) 127 [hep-th/0310232].

JHEP06(2004)001

[32] F.D.M. Haldane, *Exact jastrow-gutzwiller resonating valence bond ground state of the spin 1/2 antiferromagnetic heisenberg chain with 1/r**2 exchange*, *Phys. Rev. Lett.* **60** (1988) 635; B. Sriram Shastry, *Exact solution of an $S = 1/2$ heisenberg antiferromagnetic chain with long ranged interactions*, *Phys. Rev. Lett.* **60** (1988) 639.

[33] V.I. Inozemtsev, *On the connection between the one-dimensional $s = 1/2$ Heisenberg chain and Haldane-Shastry model*, *J. Stat. Phys.* **59** (1990) 1143.

[34] V.I. Inozemtsev, *Integrable heisenberg-van vleck chains with variable range exchange*, *Phys. Part. Nucl.* **34** (2003) 166–193 [hep-th/0201001].

[35] I.R. Klebanov, M. Spradlin and A. Volovich, *New effects in gauge theory from pp-wave superstrings*, *Phys. Lett.* **B 548** (2002) 111 [hep-th/0206221].

[36] M.R. Douglas, *Large-N gauge theory: expansions and transitions*, *Nucl. Phys.* **41** *(Proc. Suppl.)* (1995) 66 [hep-th/9409098].

[37] M. Alishahiha and A.E. Mosaffa, *Circular semiclassical string solutions on confining AdS/CFT backgrounds*, *J. High Energy Phys.* **10** (2002) 060 [hep-th/0210122]; *Semiclassical string solutions on deformed NS5-brane backgrounds and new pp-wave*, hep-th/0302005;
L. Dolan, C.R. Nappi and E. Witten, *A relation between approaches to integrability in superconformal Yang-Mills theory*, *J. High Energy Phys.* **10** (2003) 017 [hep-th/0308089];
A. Gorsky, *Spin chains and gauge/string duality*, hep-th/0308182;
D. Mateos, T. Mateos and P.K. Townsend, *Supersymmetry of tensionless rotating strings in $AdS_5 \times S^5$ and nearly-BPS operators*, *J. High Energy Phys.* **12** (2003) 017 [hep-th/0309114];
M. Schvellinger, *Spinning and rotating strings for $N = 1$ SYM theory and brane constructions*, *J. High Energy Phys.* **02** (2004) 066 [hep-th/0309161];
S. Kovacs, *On instanton contributions to anomalous dimensions in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 684** (2004) 3 [hep-th/0310193];
A. Mikhailov, *Speeding strings*, *J. High Energy Phys.* **12** (2003) 058 [hep-th/0311019];
M. Kruczenski, *Spin chains and string theory*, hep-th/0311203;
J. Stefanski, B., *Open spinning strings*, *J. High Energy Phys.* **03** (2004) 057 [hep-th/0312091];
N.-w. Kim, *Multi-spin strings on $AdS_5 \times T^{(1,1)}$ and operators of $N = 1$ superconformal theory*, hep-th/0312113;
M. Bianchi, G. Rossi and Y.S. Stanev, *Surprises from the resolution of operator mixing in $N = 4$ SYM*, *Nucl. Phys.* **B 685** (2004) 65 [hep-th/0312228].

[38] I.M. Krichever, *Elliptic solutions of the Kadomtsev-Petviashvilii equations and integrable systems of particles*, *Funct. Anal. Appl.* **14** (1980) 282.

[39] F.D.M. Haldane, Z.N.C. Ha, J.C. Talstra, D. Bernard and V. Pasquier, *Yangian symmetry of integrable quantum chains with long range interactions and a new description of states in conformal field theory*, *Phys. Rev. Lett.* **69** (1992) 2021.

[40] D. Bernard, M. Gaudin, F.D.M. Haldane and V. Pasquier, *Yang-baxter equation in spin chains with long range interactions*, hep-th/9301084.

[41] J.C. Talstra, F.D.M. Haldane, *Integrals of motion of the haldane Shastry model*, *J. Phys.* **A28** (1995) 2369, [cond-mat/9411065].

[42] D. Bernard, *An introduction to yangian symmetries*, *Int. J. Mod. Phys.* **B 7** (1993) 3517 [hep-th/9211133].

[43] D. Bernard, V. Pasquier and D. Serban, *A one-dimensional ideal gas of spinons, or some exact results on the xxx spin chain with long range interaction*, hep-th/9311013.

[44] N. Beisert, *Higher loops, integrability and the near BMN limit*, *J. High Energy Phys.* **09** (2003) 062 [hep-th/0308074].

[45] I.K. Kostov and M. Staudacher, *Multicritical phases of the $O(n)$ model on a random lattice*, *Nucl. Phys.* **B 384** (1992) 459 [hep-th/9203030].